The Invariant Charges of the Nambu-Goto String

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Content

- Diffeomorphism-invariant Quantization of string moving in d-dimensional flat background. No need for critical dimension.

- Inequivalence to Fock space methods

- Purely algebraic approach – representations?
  (work in progress)

Introduction

String action $\int_\Sigma dvol(\Sigma)$, i.e. area of the worldsheet swept out by string moving in background (here: $\mathbb{M}^d$). Hence: purely geometric object (in Riemannian background: solutions to Euler-Lagrange equations are minimal surfaces).

Canonical formalism: $H_{can} = 0$ and (primary) constraints

$$x' \cdot p \approx 0 \qquad\qquad \tfrac{1}{2}\,(p^2 + x'^2) \approx 0$$

$x^\mu : \Gamma \to \mathbb{M}^d$ parametrization, $(\tau, \sigma) \mapsto x^\mu(\tau, \sigma)$, $\tau \in \mathbb{R}_+$, $\sigma \in [0, 2\pi)$, $\,'$ derivative ∂_σ in spacelike direction ($x'^2 \leq 0$), p^μ canonical momentum, $\cdot$ Lorentz-product.

Constraints first class w.r.t. canonical Poisson bracket. Infinitesimal generators of reparametrizations.

Extended Hamiltonian linear combination of constraints

$\rightsquigarrow$ String: gauge system, 2-D-diffeomorphisms as gauge group. Toy model!

fix gauge $\equiv$ fix parametrization (e.g. choose conformal coordinates).

Fock space

Fourier decomposition of x^μ and p^μ

$\rightsquigarrow$ decomposition of constraints. Fourier modes L_n^{class} are (infinite) sums of polynomials quadratic in Fourier modes of x and p.
Poisson bracket $\rightsquigarrow$ Witt algebra

Quantization:	negative Fourier modes	$\longrightarrow$	creation op's
	positive Fourier modes	$\longrightarrow$	annihilation op's
	Poisson bracket	$\longrightarrow$	$\frac{1}{i\hbar}[\,\cdot\,,\,\cdot\,]$
	polynomial in Fourier modes	$\longrightarrow$	normal order

Witt algebra $\longrightarrow$ central extension: Virasoro algebra

$$[L_n, L_m] = (m-n)\ L_{n+m} + \frac{d}{12}(n^3 - n)\ \delta_{n,-m}$$

central charge

Physical subspace defined by $L_n|\psi\rangle = 0$, $n \geq 0$.

Works only in critical dimension $d = 26$ (needed for positive norm, closure of Lorentz algebra in light cone gauge etc...).

Enormous output, predictions etc. (e.g. massless spin 2 particle).

Problem: splitting in positive/negative Fourier modes not invariant under change of parametrization (though Fock spaces equivalent).

Are harmonic oscillators suitable? Recall Lüscher, Symanzik, Weisz 1980: Euclidean Green's function for loop equation in WKB approximation contains anharmonic terms.

Is there another way to treat diffeomorphism invariance of string theory?

The Invariant Charges

Pohlmeyer 1982: bosonic string $=$ integrable system.

Via associated system of linear equations and associated monodromy: construction of an infinite set of

diffeomorphism invariant functionals on the worldsheet
(invarariant charges)

Pohlmeyer+Rehren 1988: completeness, i.e. reconstruction of worldsheet (geometry!) from knowledge of invariant charges possible (up to rigid translations in direction of total momentum).

Invariant charges (closed string)

$$\mathcal{Z}^{\pm}_{\mu_1\ldots\mu_N} = \int\limits_{\sigma}^{\sigma+2\pi} d\sigma_1 \int\limits_{\sigma_1\leq\sigma_2<\cdots<\sigma_N\leq\sigma_1+2\pi}\!\!\!\cdots\int d\sigma_2\cdots d\sigma_N \; u^{\pm}_{\mu_1}(\tau,\sigma_1)\cdots u^{\pm}_{\mu_N}(\tau,\sigma_N)$$

auxiliary variables: right/left movers $u^{\pm}_{\mu} = p_\mu \pm x'_\mu$.

- Invariance: Poisson-commute (strongly!) with constraints.
- No gauge fixing required in definition.
- Examples: $\mathcal{P}_\mu = \oint d\sigma u^{\pm}_{\mu}(\tau,\sigma) = \oint d\sigma p_\mu(\tau,\sigma)$, Pauli-Lubanski vector... higher orders (large N) no such simple interpretation – recall completeness!
- Form a Poisson algebra (i.e. close under Poisson bracket and multiplication). Right and Left mover part commute with each other. $\mathcal{P}_\mu$ are central.
- Covariance under proper orthochronous Lorentz transform.

The Poisson Algebra Invariant Charges

- graded, $\mathfrak{h} = \bigoplus_{l=0}^{\infty} \mathfrak{h}^l$. Dimension of each stratum $\mathfrak{h}^l$ (as vectorspace) finite (depends on d and l). Physical unit of elements of $\mathfrak{h}^l$: action^{l+1}.

$$\{\mathfrak{h}^{l_1}, \mathfrak{h}^{l_2}\} \subset \mathfrak{h}^{l_1+l_2} \ , \qquad \mathfrak{h}^{l_1} \cdot \mathfrak{h}^{l_2} \subset \mathfrak{h}^{l_1+l_2+1}$$

- Technical problem: algebraic dependences between invariant charges, i.e. certain linear combinations of (multiple) Poisson brackets and products are 0 (relations).

- Number of independent (generating) relations in each stratum is known (depends on d). Explicit form also depends on d.

- Aim: presentation of $\mathfrak{h}$ in terms of generators and relations.

- Relations calculated explicitly in lower strata (< 10) for $d = 3, 4$, massive and massless case ($\mathcal{P}^2 > 0$ and $\mathcal{P}^2 = 0$)

[Pohlmeyer, Rehren, Happle, Schneider ....]

Invariant Quantization

- 1-1 correspondence classical $\leftrightarrow$ quantum generators
- Graded algebra $\mathfrak{h} \rightarrow$ filtered algebra $\hat{\mathfrak{h}}$ (allow for quantum corrections). Physical unit of elements of level $\hat{\mathfrak{h}}^l$ (l from 0 to ∞) is $\hbar^{l+1}$.
- Relations: Poisson brackets $\rightarrow$ commutators, multiplication noncommutative.

  Admit quantum corrections, i.e. (commutators and products of) elements of lower strata, multiplied with appropriate powers of $\hbar$ and some at this stage free complex parameters $\rightarrow$ quantum relations
- Quantum corrections (possibly uniquely) fixed by structural similarity conditions (e.g. number of independent relations should remain unchanged)

  have been computed explicitly in lower strata (massive string, $d = 3, 4$)
- Calculation of quantum corrections $=$ consistency check of programme. Since works stratum by stratum, no general proof.

Breakthrough:

Classical $\mathfrak{h}$ again: (infinite) set of functionals $\mathcal{R}^t_{\mu_1\cdots\mu_N}$ on worldsheet with only *linear* dependences: not themselves invariant under reparametrizations, but invariant charges are polynomials in $\mathcal{R}^t_{\mu_1\cdots\mu_N}$'s.

Multiplicative and Poisson structure of $\mathfrak{h}$ reproduced.

Meusburger+Rehren 02: quantize Poisson algebra of $\mathcal{R}^t_{\mu_1\cdots\mu_N}$'s $\rightsquigarrow$ consistent quantization of $\mathfrak{h}$ (to all orders) fulfilling structural similarity conditions.

$\Rightarrow$ there is a consistent quantization: no critical dimension!

Explicitly computed quantum corrections for massive string in $d = 4$ are reproduced.

Quantization on Fock space?

Classically: Fourier decomposition of $u^\pm \rightsquigarrow \mathcal{Z}^\pm$ infinite sums of polynomials of *arbitrarily high degree* in Fourier modes α_μ^n, β_μ^n

$$\mathcal{Z}^+_{\mu_1\ldots\mu_N} = \sum_{n_1=-\infty}^{\infty} \cdots \sum_{n_N=-\infty}^{\infty} \alpha_{\mu_1}^{n_1} \cdots \alpha_{\mu_N}^{n_N} \;\; C^{(N)}_{n_1,\ldots,n_N}$$

Classical Poisson structure reproduced. In particular, generating relations ok $\surd$

Quantization:

annihilation/ creation op's. Normal order $\rightsquigarrow$ $: \mathcal{Z}^\pm_{\mu_1\ldots\mu_N} :$

[DB, JMP 04]

Relations?

In classical relation: Poissonbrackets $\rightarrow$ commutators. Insert : $\mathcal{Z}^{\pm}_{\mu_1 \dots \mu_N}$:

Evaluate commutators by application of derivation rule, bring result into normal order $\rightarrow$ correction terms.

In $d = 4$ already in second stratum: correction terms do not correspond to invariant charges (anomalies). Typical term: $\sum_{n>0} \frac{1}{n} \alpha_i^{-n} \alpha_i^n$

Non-invariance: take classical counterpart (α Fourier modes). Then

- direct proof: even Möbius transform does not leave this invariant.
- alternative: term does not Poisson commute with classical constraints.

Generating quantum relations not reproduced ($d = 4$)!

Not even on physical subspace: anomalies do not annihilate physical vectors.

Independence of dimension d?

Slight chance: generating relations in $d = 26$? (admit suitable non-invariant quantum corrections in quantized invariants?). Tedious to calculate even classical relations (future project). However:

- Generating relations from $d = 4$ remain true as *classical identities* in arbitrary dimension $d \Rightarrow$ anomalies appearing after naive quantization on Fock space still mean: algebra does not close!

  For simple commutators proved: change of ordering prescription does not help.

- Moreover, $[L^n, :\mathcal{Z}^{\pm}_{\mu_1 \dots \mu_N}:] \neq 0$ even on physical subspace and for any dimension d. Again, rhs not in the algebra.

  Mechanism similar to $[M_{i-}, M_{j-}]$ in lightcone gauge. But: no simple solution (e.g. $d = 26$), since $\mathcal{Z}$ not only quadratic in α!

Representations

- Thiemann 2004: an invariant quantization of invariant charges via representation using diff-invariant LQG state (does not reproduce quantum relations). No critical dimension!

- Schreiber 2004: *classical* Poisson algebra of invariant charges in terms of DDF operators. Quantization? (problem, in particular, with restoring Poincaré-invariance).

- Finite dimensional representations (not faithful): only a limited number of generators represented nontrivially

  VERY promising!!!

Outlook

- Continue working on (finite) representations (long term goal: find full representation of quantum algebra $\hat{\mathfrak{h}}$ with constraint operators as Casimirs).
- Scattering: encoded in algebraic relations between algebras of the respective three legs – analyze using (finite) representations

Future:

- Backgrounds other than $\mathbb{M}^d$? (cp. Beisert, Staudacher et al)