Linearized Gravity as the Rosetta Stone of Quantization Techniques

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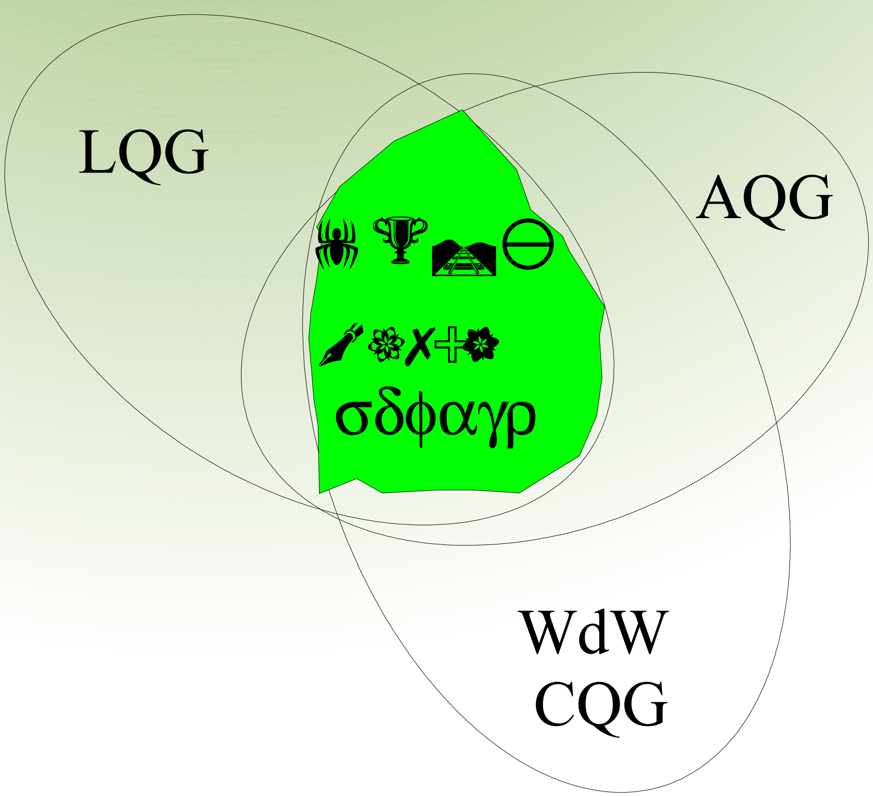
On the obvious common ground...

- **Classical limit**
  - General relativity
  - Weak field

- **Quantum models**
  - 2+1
  - Sourcefree EM
  - **Linearized gravity**



- Here we can easily learn **new** techniques from one another!

Overview

- Projection operator quantization
  - Introduction
  - Application to linearized gravity
- Reproducing kernel Hilbert space
  - Canonical CS representation
  - CS path integral
- Summary
- Challenges

Classical considerations

- Constrained, phase space action

$$I[p,q] = \int dt\, [p_a \dot{q}^a - H(p,q) + \lambda^\alpha \phi_\alpha(p,q)]$$

- Canonical symplectic manifold

$$\{q^a, p_b\} = \delta^a_b$$

- General constraints possible; simplest example

$$\{\phi_a(p,q), \phi_b(p,q)\} = c_{ab}{}^c\, \phi_c(p,q)$$

$$\{H(p,q), \phi_a(p,q)\} = h_a{}^b\, \phi_b(p,q)$$

Projection operator quantization

- Promotion to operators (ala Dirac)

$$\{q^j, p_k\} = \delta^j_k \mapsto [Q^j, P_k] = i\hbar\, \delta^j_k$$

$$\phi_m(p,q) = 0 \mapsto \Phi_m(P,Q)|\psi_P\rangle = 0$$

- Physical Hilbert space populated by states of the form

$$|\psi_P\rangle \in \mathfrak{H}_P \subset \mathfrak{H} \qquad \mathbb{E}|\psi\rangle = |\psi_P\rangle$$

- Projection operator obeys the relations

$$\mathbb{E}^2 = \mathbb{E} \qquad \mathbb{E}^\dagger = \mathbb{E}$$

- Formally, it may be represented by

$$\mathbb{E}\left[\Sigma\, \Phi_a^2 \le \delta(\hbar)^2\right] = \int_{-\infty}^{\infty} d\lambda\, \exp\left[-i\lambda\ \Sigma\, \Phi_a^2\right] \frac{\sin(\delta^2\lambda)}{\pi\lambda}$$

Reproducing kernels

- Operators uniquely determined by CS matrix elements

$$\langle p', q'|\mathbb{E}|p, q\rangle$$

- Define the reproducing kernel (RK)

$$\langle\langle p', q'|p, q\rangle\rangle \equiv \lim_{\delta\to 0} \frac{\langle p', q'|\mathbb{E}[X^2 < \delta^2]|p, q\rangle}{\langle 0|\mathbb{E}[X^2 < \delta^2]|0\rangle}$$

- Elements of the physical Hilbert space

$$\psi(p, q) = \sum_{n=1}^{N} \alpha_n \, \langle\langle p, q|p_n, q_n\rangle\rangle$$

- Inner product

$$(\psi, \phi) = \sum_{m,n=1}^{M,N} \alpha_n^* \beta_m \, \langle\langle p_n, q_n|p_m, q_m\rangle\rangle$$

Linearized canonical gravity

- The 3-metric and it's conjugate momentum are perturbed
- ADM action is dynamical in the perturbations

$$I[p,h] = \int dt \int d^3x \left[p_{ab}\dot{h}^{ab} - N^0 H_0 \quad - N^i H_i \quad \right]$$

- Poisson algebra

$$\{h_{ab}(x), p^{cd}(x')\} = \delta^c_{(a}\delta^d_{b)}\delta(x - x')$$

- Linearized constraints

$$\begin{aligned} H_0^{(1)}(x) &= -R^{(1)} = h_{ij,ij} - h_{ii,jj} \\ H_i^{(1)}(x) &= -2p_i{}^k{}_{,k} \end{aligned}$$

Quantum linearized gravity

- Commutation relations

$$\left[\hat{h}_{ab}(x), \hat{p}^{\dagger cd}(x')\right] = i\delta^c_{(a}\delta^d_{b)}\delta^3(x - x')$$

- Useful orthogonal expansion

$$h_{ab}(\mathbf{k}) \rightarrow \mathbf{h}^{TT}_{ab}(\mathbf{k}) + \mathsf{h}^{T}_{ab}(\mathbf{k}) + \mathfrak{h}^{L}_{ab}(\mathbf{k})$$

- Simplification of the constraints

$$\begin{aligned} \hat{H}_0(\mathbf{k}) &= k^2\, \hat{\mathsf{h}}^T(\mathbf{k}), \\ \hat{H}_a(\mathbf{k}) &= \left[k^2 \hat{\mathfrak{p}}^L(\mathbf{k})\right]_a \end{aligned}$$

A projection operator for linearized gravity

- One projection operator applies per point in momentum space

$$\mathbb{E}\left[\sum_a \Phi_a^2 \leq \epsilon^2\right] = \mathbb{E}\left[|\hat{\mathfrak{h}}^T|^2 + |\hat{\mathfrak{p}}^L|^2 \leq \delta_{\mathbf{k}}^2\right]$$
$$= \int_{-\infty}^{\infty} d\lambda \, \exp\left[-i\lambda\left(|\hat{\mathfrak{h}}^T|^2 + |\hat{\mathfrak{p}}^L|^2\right)\right] \frac{\sin(\delta_{\mathbf{k}}^2 \lambda)}{\pi\lambda}$$

- Commutation of the constraints allows for a split in the projection operator

$$\mathbb{E} \equiv \mathbb{E}^T\left[|\hat{\mathfrak{h}}^T|^2 \leq \frac{\delta_{\mathbf{k}}^2}{2}\right] \mathbb{E}^L\left[|\hat{\mathfrak{p}}^L|^2 \leq \frac{\delta_{\mathbf{k}}^2}{2}\right]$$
$$= \int_{-\infty}^{\infty} d\mu[\xi] \, \exp\left\{-i\xi|\hat{h}^T|^2\right\} \int_{-\infty}^{\infty} d\mu[\zeta] \, \exp\left\{-i\zeta\,|\hat{\mathfrak{p}}^L|^2\right\}$$

- This means a split in the each reproducing kernel

Canonical coherent states

- The canonical CS may be written as

$$
\begin{aligned}
|\mathbf{p},\mathbf{h}\rangle^{TT} &= e^{-i\int du(k)\ \mathbf{h}^*_{ab}(\mathbf{k})\hat{\mathbf{p}}^{TT}_{ab}(\mathbf{k})}e^{i\int du(k)\ \mathbf{p}^*_{ab}(\mathbf{k})\hat{\mathbf{h}}^{TT}_{ab}(\mathbf{k})}|0\rangle \\
|\mathsf{p},\mathsf{h}\rangle^{T} &= e^{-i\int du(k)\ \mathsf{h}^*(\mathbf{k})\hat{\mathsf{p}}^{T}(\mathbf{k})}e^{i\int du(k)\ \mathsf{p}^*(\mathbf{k})\hat{\mathsf{h}}^{T}(\mathbf{k})}|0\rangle \\
|\mathfrak{p},\mathfrak{h}\rangle^{L} &= e^{-i\int du(k)\ \mathfrak{p}^*(\mathbf{k})\cdot\hat{\mathfrak{h}}^{L}(\mathbf{k})}e^{i\int du(k)\ \mathfrak{h}^*(\mathbf{k})\cdot\hat{\mathfrak{p}}^{L}(\mathbf{k})}|0\rangle
\end{aligned}
$$

- For such states, the overlap appears as

$$\langle\mathfrak{p},\mathfrak{h}|\mathfrak{p}',\mathfrak{h}'\rangle = \exp\left\{-\int\frac{d^3k}{(2\pi)^{3/2}}\ \left[|\mathfrak{z}(\mathbf{k})|^2/2+|\mathfrak{z}'(\mathbf{k})|^2/2-\mathfrak{z}^*(\mathbf{k})\cdot\mathfrak{z}'(\mathbf{k})\right]\right\}$$

How the longitudinal degrees of freedom decouple...

- The longitudinal reproducing kernel becomes in the limit

$$\lim_{\substack{N\to\infty \\ \delta_{\mathbf{k}}\to 0}} \frac{\langle\, \mathfrak{p}, \mathfrak{h} | \mathbb{E}_N^L | \mathfrak{p}', \mathfrak{h}' \rangle}{\langle 0 | \mathbb{E}_N^L | 0 \rangle} \to \mathcal{F}_0^*[\mathfrak{p}, \mathfrak{h}]\, \mathcal{F}_0[\mathfrak{p}', \mathfrak{h}']$$

where

$$\mathcal{F}_0[\mathfrak{p}, \mathfrak{h}] \equiv \exp \int \frac{d^3k}{(2\pi)^{3/2}} \left[-\frac{\|\mathfrak{p}(\mathbf{k})\|^2}{4\omega(\mathbf{k})} - \frac{\omega(\mathbf{k})\|\mathfrak{h}(\mathbf{k})\|^2}{4} \right]$$

- Calculation of the transverse RK yields a similar expression
- Therefore each of the longitudinal and transverse Hilbert spaces reduces to a **one dimensional** Hilbert space

An alternative RK representation

- With a suitable measure, the projection operator may be expressed as

$$\mathbb{E} = \int \mathcal{D}R(N^a, N^0)\, \mathbf{T}\, e^{-i\int d^4x[N^a\ \hat{H}_a(\mathbf{x}) + N^0\ \hat{H}_0(\mathbf{x})]}$$

- In the sense of coherent state matrix elements, the RK becomes

$$\begin{aligned} RK &\equiv \lim_{\nu\to\infty} N_\nu \int \exp\left\{-i\int d^4x[\dot{p}^{ab}h_{ab} + N^a\ H_a + N^0\ H_0]\right\} \\ &\times \exp\left\{-\frac{1}{2\nu}\int d^4x[\dot{p}^{ab}\dot{p}_{ab} + \dot{h}_{ab}\dot{h}^{ab}]\right\}\prod_{x,t}\prod_{a<b} dp^{ab}dh_{ab}\mathcal{D}R \end{aligned}$$

Summary of Results Using Projection Operator Quantization

- Kinematical Hilbert space reduced to physical Hilbert space

$$
\begin{aligned}
\mathbb{E}\,\mathfrak{H} &= \mathfrak{H}_{TT} \otimes \mathbb{E}\left[\mathfrak{H}_L \otimes \mathfrak{H}_T\right] \\
&= \mathfrak{H}_{TT} \otimes \mathbb{E}^T\,\mathfrak{H}_T \otimes \mathbb{E}^L\,\mathfrak{H}_L \\
&= \mathfrak{H}_{TT} \otimes \mathbb{C} \otimes \mathbb{C} \approx \mathfrak{H}_{TT}
\end{aligned}
$$

- A **non-compact gauge group** was quantized

- **No gauge choices were made in quantizing the theory!**

What the Rosetta Stone really says…

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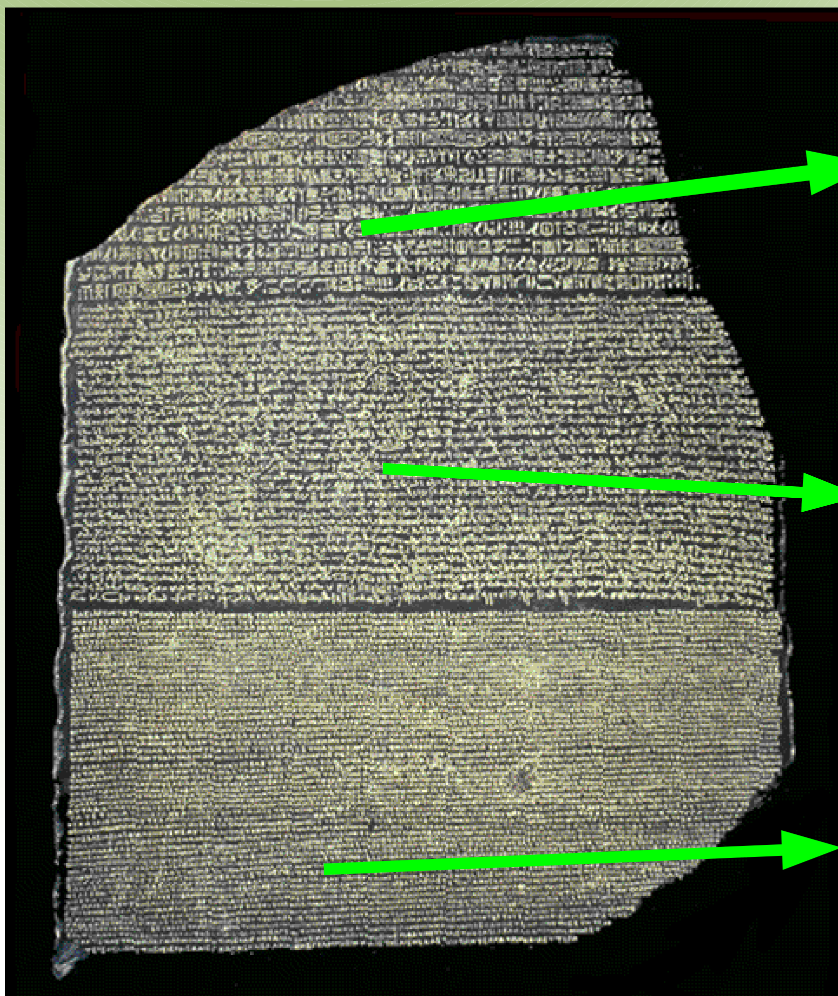

- CQG: configuration space

$$\Psi_0 = \mathcal{N} \exp\left[-\frac{1}{2}\int d^3k \;\; \omega(\mathbf{k}) \;\; \left|h^{TT}_{mn}(\mathbf{k})\right|^2\right]$$

- Loops: U(1) x U(1) x U(1)

$$\Phi_0^L = \sum_{\boldsymbol{\alpha},q} \exp\left\{\frac{-i\gamma}{4}\int d^3x \; G_{ab}^{(r);\boldsymbol{\alpha}\,*}(\mathbf{x}) \; X^{ab}_{(r);\boldsymbol{\alpha}}(\mathbf{x})\right\} \langle \boldsymbol{\alpha}, q|$$

- **RK: phase space**

$$\langle \mathbf{p}, \mathbf{h}|0\rangle^{TT} = \exp\left[-\int \frac{d^3k}{4\omega(\mathbf{k})}\left(|\mathbf{p}_{mn}(\mathbf{k})|^2 + \omega(\mathbf{k})^2|\mathbf{h}_{ab}(\mathbf{k})|^2\right)\right]$$

Future challenges

- **Development issues**
  - What happens to delta when describing interacting QFT's?
  - Application of the projection operator to 2+1 GR
  - More mathematical rigor
- **Can the projection operator be used to constrain linear loops?**
  - Affine coherent states
  - Projection operator in the complexifier CS machinery