Anisotropic homogeneous models in loop quantum cosmology

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$\boxed{\text{Outline}}$

1. Loop quantum cosmology

2. Generating function techniques

3. Complete solutions

4. Results

   - Full Bianchi I
   - Bianchi I LRS
   - Self-adjoint Schwarzschild interior (preliminary)

5. Conclusions, future work

$\boxed{\text{Loop quantum cosmology}}$

Symmetry-reduced form of loop quantum gravity (LQG)

- Kinematics of LQG well understood

- Dynamics (Hamiltonian constraint of theory) is where the difficulty is

- By analogy with minisuperspace models, what can we learn from simplified cosmological systems?

- What physical properties can we deduce?

For anisotropic, homogeneous models in loop quantum cosmology, the constraint is a partial *difference* equation, instead of a differential equation

- As an example, we look in detail at vacuum Bianchi I LRS (local rotational symmetry), where the Hamiltonian constraint becomes

$$2d_2(m)[t_{m+1,n+1} - t_{m+1,n-1} - t_{m-1,n+1} + t_{m-1,n-1}]$$
$$+d(n)[t_{m+2,n} - 2t_{m,n} + t_{m-2,n}] = 0$$

for wave function coefficients $t_{m,n}$ and parameters m, n, with

$$d(n) = \sqrt{\left|1 + \frac{1}{2n}\right|} - \sqrt{\left|1 - \frac{1}{2n}\right|}$$

and

$$d_2(m) = \frac{1}{m} \qquad m \neq 0$$

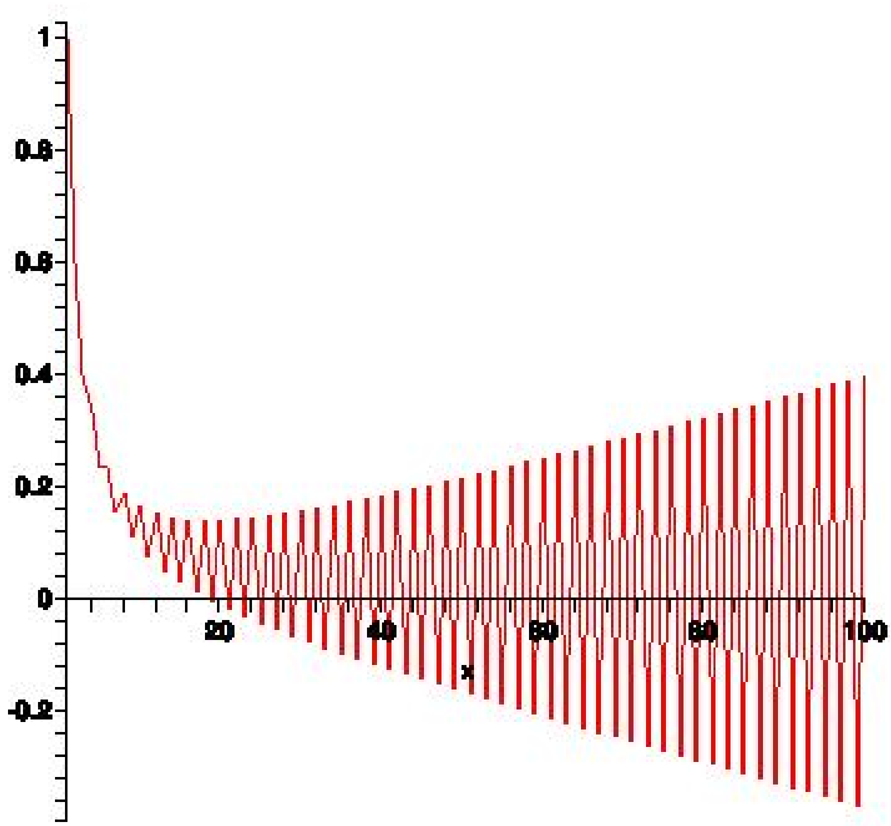
Later, we will use the fact that

$$d(n) = \frac{1}{2n} + O(n^{-3}) \simeq \frac{d_2(n)}{2}$$

when $n > 1/2$.

In general, solving recursion relations with generic initial data gives sequences that alternate sign with increasing parameters $(m, n, \ldots)$

$$m(a_{m+1} - a_{m-1}) + 2a_m = 0 \qquad [m > 0]$$



Bianchi I LRS constraint:

$$2d_2(m)[t_{m+1,n+1} - t_{m+1,n-1} - t_{m-1,n+1} + t_{m-1,n-1}]$$
$$+d(n)[t_{m+2,n} - 2t_{m,n} + t_{m-2,n}] = 0$$

To simplify this recursion relation, we notice that it is separable into two equations:

$$a_{m+1} - a_{m-1} = (2\lambda/m)a_m$$
$$b_{n+1} - b_{n-1} = -\lambda d(n) b_n$$

Because $d(n) \simeq 1/2m$, these are essentially the same equation (up to scaling of the separation parameter λ).

We insist on "pre-classicality" – wave functions are smooth at large values of the parameters; for example, in a one-parameter sequence a_m,

$$(a_m - a_{m-1}) \to 0 \qquad m \to \infty$$

Generating function techniques

Instead of dealing with a sequence $\{a_m\}$, write these as coefficients of a generating function $F(x)$

$$\{a_m\} \quad \Leftrightarrow \quad F(x) = \sum_{m=0}^{\infty} a_m x^m$$

Operations on $\{a_m\}$ map to operations on $F(x)$:

$$\{a_{m+1}\} \quad \Leftrightarrow \quad \frac{F(x) - a_0}{x}$$

$$\{m a_m\} \quad \Leftrightarrow \quad x \, \frac{dF(x)}{dx}$$

The generalization for an arbitrary number of parameters m_i is obvious...

We write the Hamiltonian constraint for wave functions as a differential equation for the generating functions $F(x_i)$ of our sequence

In general, when you solve the PDE for the generating function $F(x_i)$, it has singularities at various points

- Some singularities are "bad", such as $(1+x)^{-1}$, where the coefficients $\{a_m\}$ of the series expansion oscillate:

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots$$

- Others are "good", such as $(1-x)^{-1}$, where the coefficients $\{a_m\}$ are constant

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

  Note this has to be a simple pole to avoid coefficients increasing without limit, since

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$

Pre-classicality, asymptotic behavior of sequences $\Leftrightarrow$ conditions on poles of generating functions

Solving the recursion relation

$$a_{m+1} - a_{m-1} = (2\lambda/m)a_m$$

turns out to be equivalent (for $m > 0$) to solving the ODE

$$\frac{d}{dx}[(1-x^2)G(x)] - 2\lambda G(x) = a_0$$

where

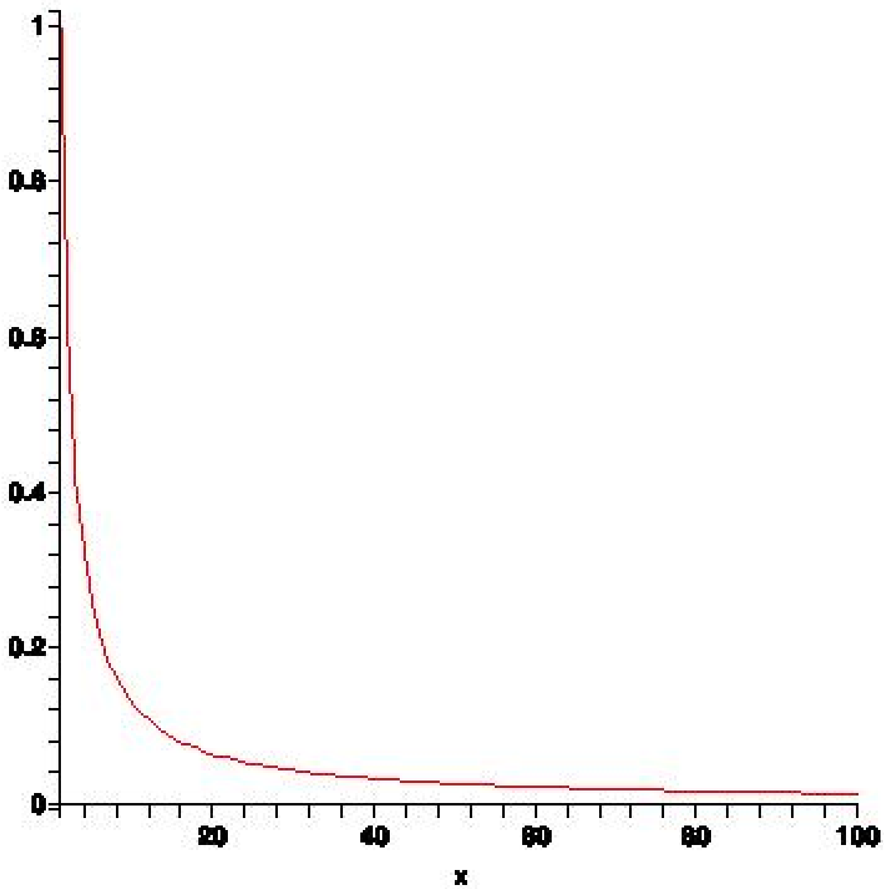
$$F(x) = a_0 + xG(x)$$

Because of the $(1-x^2)$ term multiplying $dG(x)/dx$, the solution $G(x)$ can have poles at $x = \pm 1$.

The generating function for the relation with $\lambda = -1$ will have a double pole at $x = -1$; requiring

$$[(1 + x)^2 F(x)]_{x=-1} = 0$$

gives a condition on the ratio a_1/a_0 of the initial data of the sequence



$$\boxed{\begin{array}{c}\text{Complete solutions} \\ \text{(gr-qc/0506024)}\end{array}}$$

The advantage of using generating functions is that we can obtain the values a_m for *any* m, not just integers.

For example, with $a_0 = 0$ and $\lambda > 0$, we have that the generating function of the sequence is

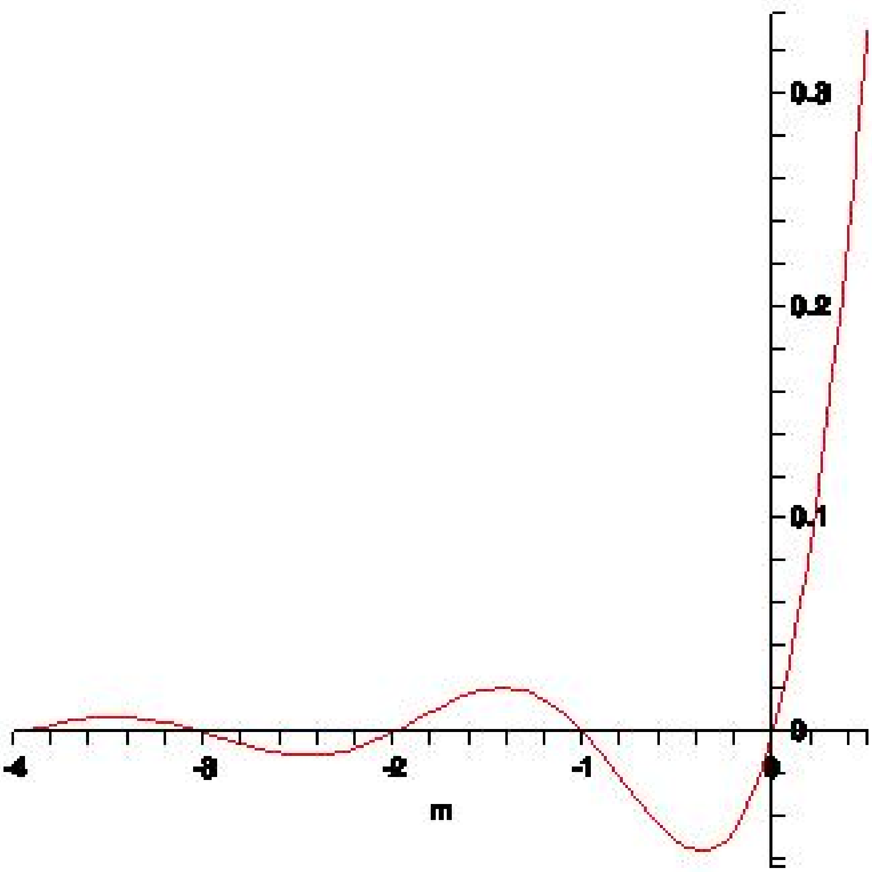
$$F(x) = a_1 x (1 + x)^{\lambda - 1} (1 - x)^{-\lambda - 1}$$

The coefficients of the Taylor series give the values a_m:

$$a_m = a_1 \sum_{j=0}^{m-1} (-1)^j \binom{\lambda - 1}{m - j - 1} \binom{-\lambda - 1}{j}$$

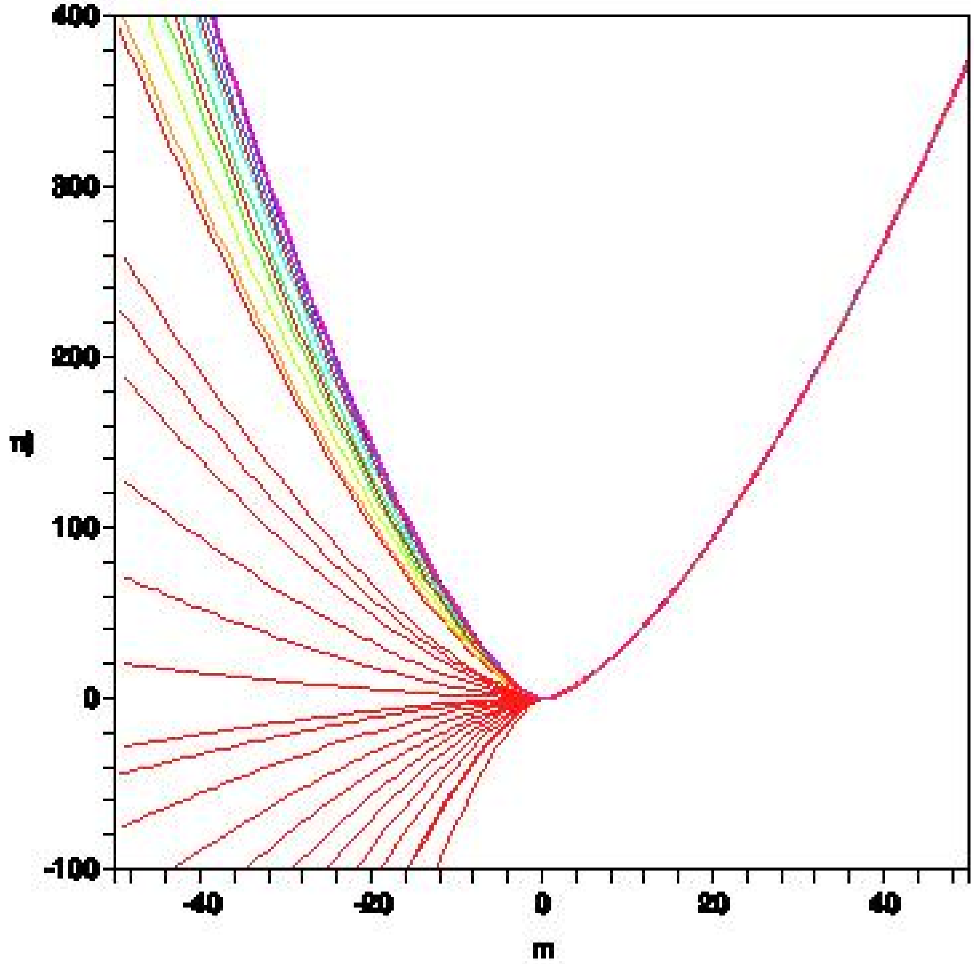
By using hypergeometric functions, this can be defined for all real values of m.

Plot of a_m when $\lambda = 3/2$



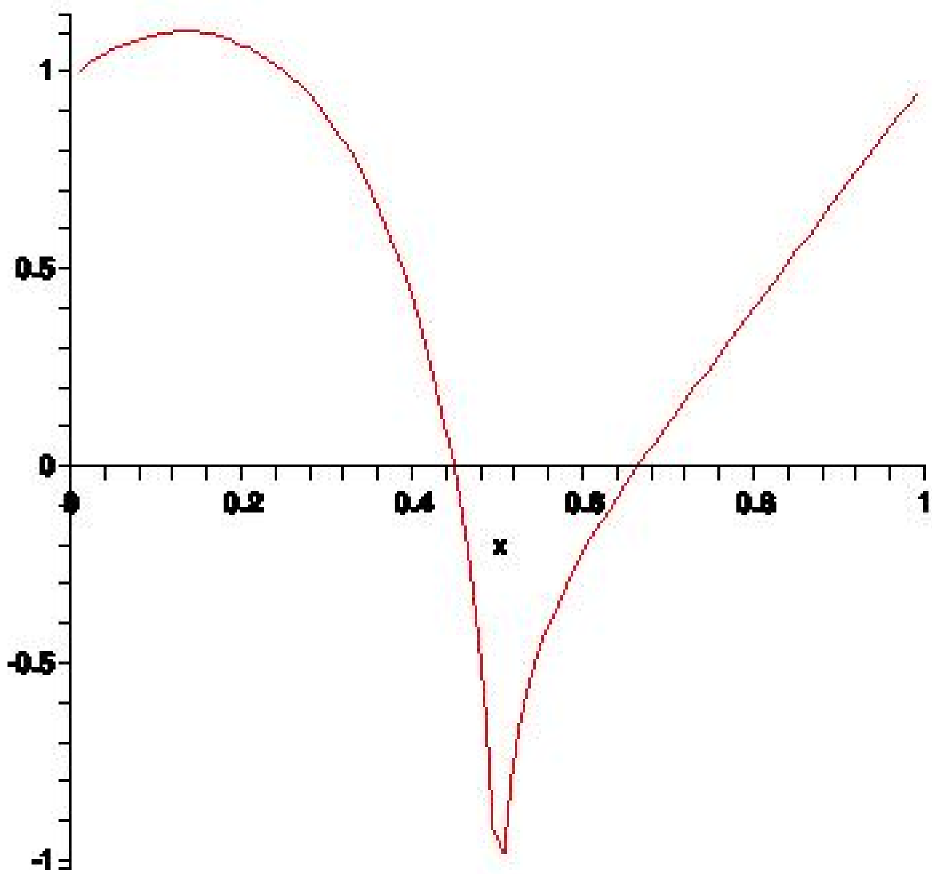
(using $1/m$ in the recursion relation)

When $d(n)$ is used in the recursion relation, sequences must be numerically evolved from large m; again choosing $\lambda = 3/2$...



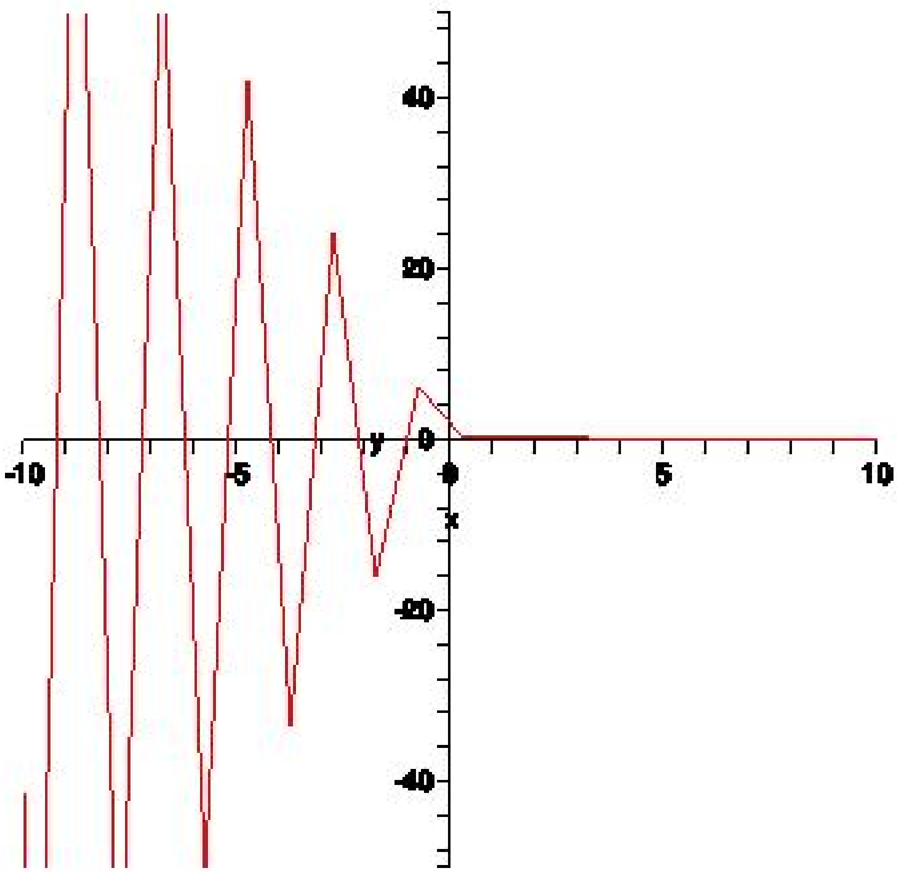
each line on the LHS represents a non-integer b, where $m = n + b$, (n an integer).

The scaling of the sequence varies as a function of the non-integer "remainder" b...



Plot of the ratio a_M/a_{-M} for large M versus b (for $\lambda = 3/2$)

For $\lambda < 0$, solutions are more limited – either there are no pre-classical solutions (for $a_0 = 0$) or the sequence is pre-classical only on one side ($a_0 \neq 0$).



Plot with $\lambda = -1, m = n + 3/4$

Results: Full Bianchi I (gr-qc/0501016)

Classical metric given by

$$ds^2 = -dt^2 + \sum_{i=1}^{3} a_i^2(t) dx_i^2$$

for scale factors $a_i(t)$

Hamiltonian constraint for vacuum case in LQC:

$$d(m_1)[t_{m_1,m_2+1,m_3+1} - t_{m_1,m_2-1,m_3+1} - t_{m_1,m_2+1,m_3-1} + t_{m_1,m_2-1,m_3-1}] + \text{(cyclic)} = 0$$

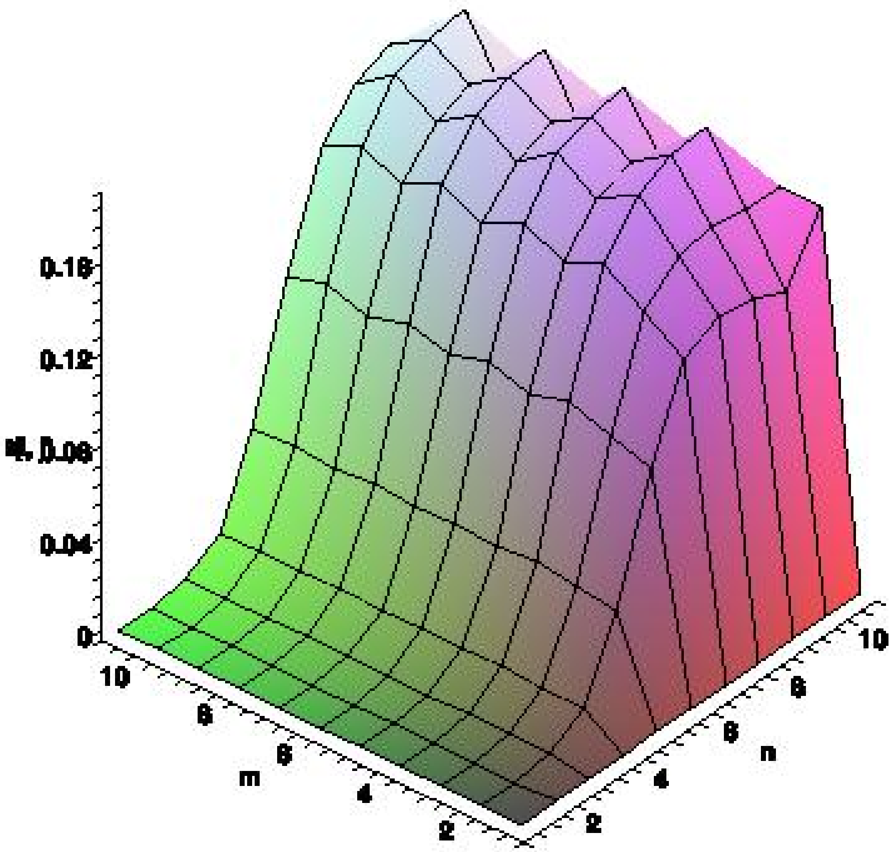
This can be written as a product of three separable sequences, satisfying the relations discussed previously, with

$$\lambda_1\lambda_2 + \lambda_1\lambda_3 + \lambda_2\lambda_3 = 0$$

Since pre-classicality requires $\lambda_k > 0$, there are no pre-classical solutions *at all* for this recursion relation!

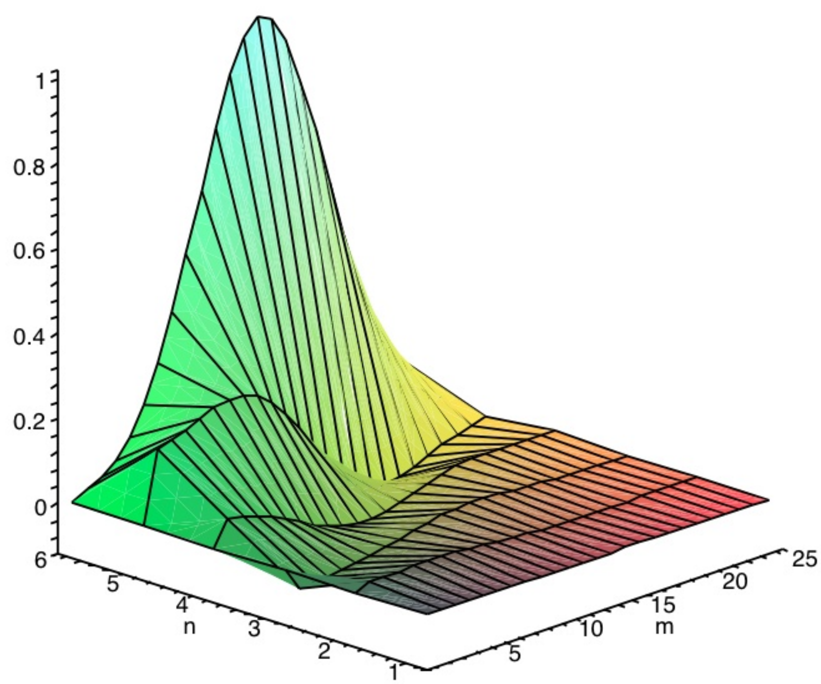
Results: Bianchi I LRS
(gr-qc/0405126)

Two of the three scale factors $a_i(t)$ in the metric on the previous slide are the same



Slope goes to zero for $m \gg 1$; essentially no variation in n direction

Using numerical techniques adapted to solving these difference equations (Connors and Khanna, gr-qc/0509081), one can find solutions when a cosmological constant is added:



For large m, n, these numerical solutions should be matchable to the semi-classical solution of the WdW equation

Results: Self-adjoint Schwarzschild interior (preliminary)

The recursion relation away from the horizon ($m \geq 2$) is given by (Ashtekar and Bojowald, gr-qc/0509075)

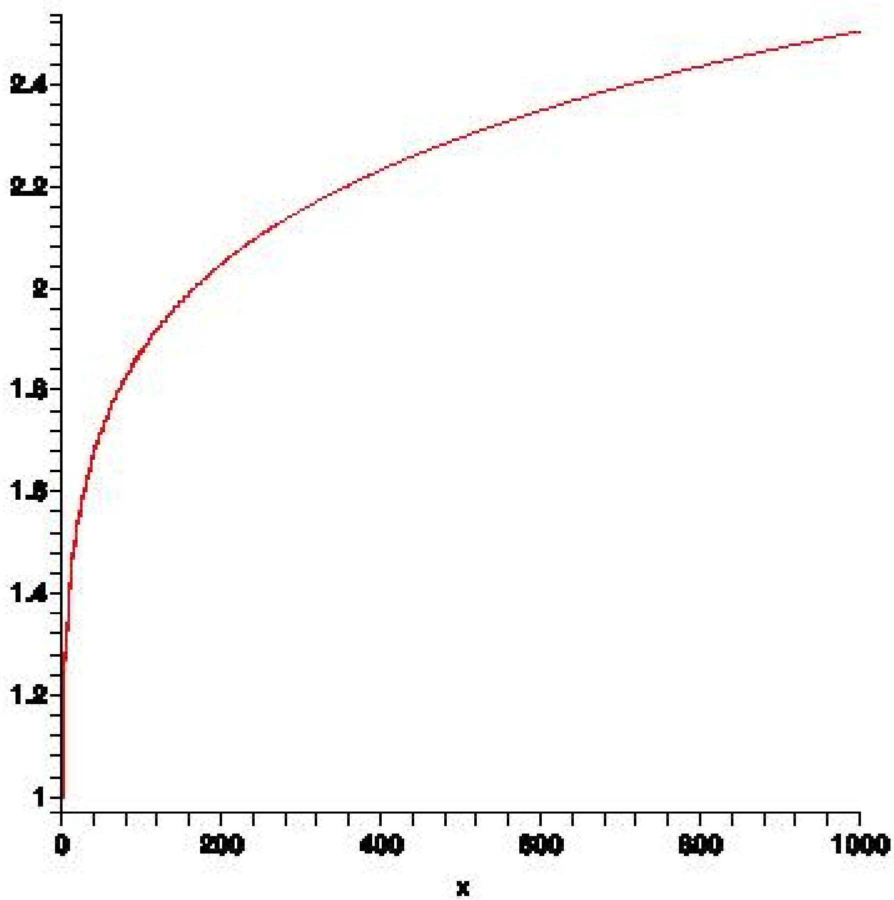
$$
\begin{gathered}
(\sqrt{|n+1|}+\sqrt{|n|})(s_{m+1,n+1}-s_{m-1,n+1}) \\
+(\sqrt{|n-1|}+\sqrt{|n|})(s_{m-1,n-1}-s_{m+1,n-1}) \\
+(\sqrt{|n+1/2|}-\sqrt{|n-1/2|})\big[(n+1)s_{m+2,n} \\
-\kappa n s_{m,n}+(n-1)s_{m-2,n}\big]=0
\end{gathered}
$$

The constant κ depends on the Barbero-Immirzi parameter γ and a quantum ambuigity δ

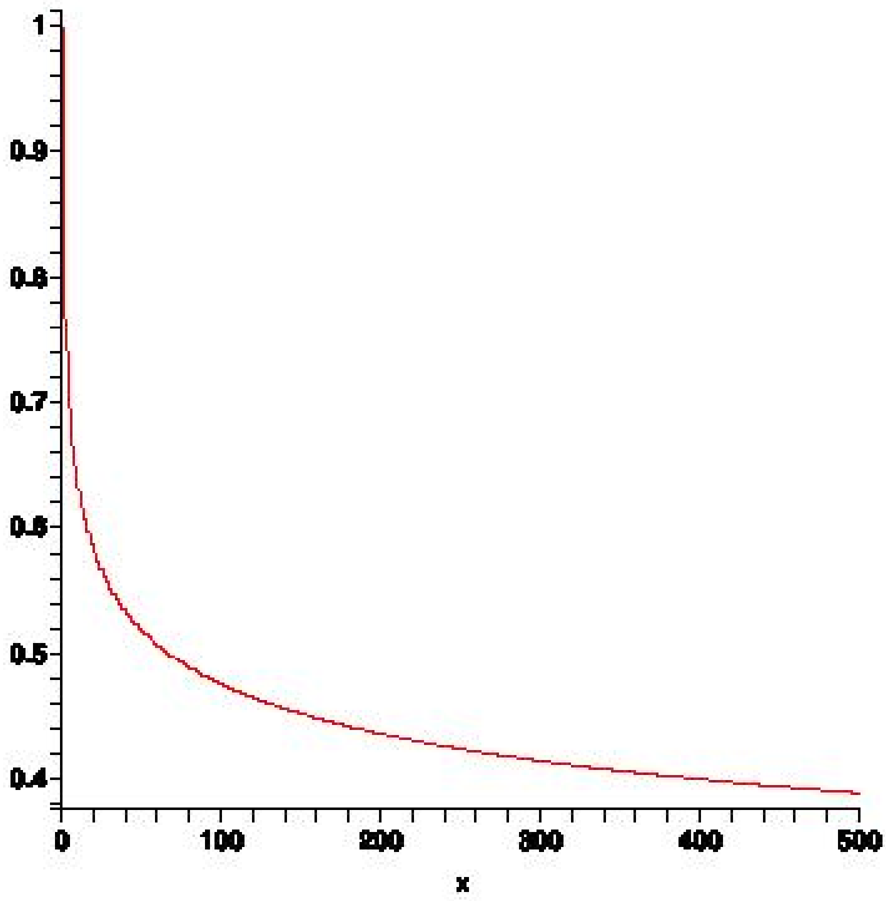
Again, this is separable into two relations, which are more complicated that those for Bianchi I

For the "time" sequence b_n which passes through the singularity $n = 0$, there is no restriction on the value of the separation parameter λ...

When $\lambda < -2$, initial data is free of restrictions and the solution grows without bound:



When $\lambda > -2$, the ratio b_1/b_0 is fixed to ensure no oscillations far from the origin



For the ”spatial” a_m sequences, the generating function has the polynomial $(x^4 - \kappa x^2 + 1)$ in front of the derivative dG/dx. Since $\kappa \geq 0$, this means there are poles at $x = \pm x_0, \pm 1/x_0$.

One of these must be avoided to prevent oscillations at large m; another will cause the sequence to grow without limit (choose $a_m \to 0$ as $m \to \infty$ as boundary condition).

Since there are *four* initial values $(a_0, \ldots, a_3)$, with at most two to fix, it seems that we again have freedom in our choices for all λ

Conclusions, future work

- Generating function methods are useful in finding the space of pre-classical solutions for the discrete Hamiltonian constraint in LQC

- Against expectations, pre-classical solutions for vacuum anisotropic models studied so far are non-existent (Bianchi I) or very limited (Bianchi I LRS)

- It appears there is sufficient freedom in the Schwarzschild interior to build up generic wave forms at the horizon of the black hole

- Further avenues to explore:

  - addition of matter (e.g. scalar field)

  - physical wave functions via group averaging