Anisotropic homogeneous models in loop quantum cosmology

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1. Loop quantum cosmology

2. Generating function techniques

3. Complete solutions

4. Results
   - Full Bianchi I
   - Bianchi I LRS
   - Self-adjoint Schwarzschild interior (preliminary)

5. Conclusions, future work
Loop quantum cosmology

Symmetry-reduced form of loop quantum gravity (LQG)

- Kinematics of LQG well understood

- Dynamics (Hamiltonian constraint of theory) is where the difficulty is

- By analogy with minisuperspace models, what can we learn from simplified cosmological systems?

- What physical properties can we deduce?
For anisotropic, homogeneous models in loop quantum cosmology, the constraint is a partial *difference* equation, instead of a differential equation

- As an example, we look in detail at vacuum Bianchi I LRS (local rotational symmetry), where the Hamiltonian constraint becomes

\[
2d_2(m)[t_{m+1,n+1} - t_{m+1,n-1} - t_{m-1,n+1} + t_{m-1,n-1}] \\
+ d(n)[t_{m+2,n} - 2t_{m,n} + t_{m-2,n}] = 0
\]

for wave function coefficients \( t_{m,n} \) and parameters \( m, n \), with

\[
d(n) = \sqrt{|1 + \frac{1}{2n}|} - \sqrt{|1 - \frac{1}{2n}|}
\]

and

\[
d_2(m) = \frac{1}{m} \quad m \neq 0
\]

Later, we will use the fact that

\[
d(n) = \frac{1}{2n} + O(n^{-3}) \sim \frac{d_2(n)}{2}
\]

when \( n > 1/2 \).
In general, solving recursion relations with generic initial data gives sequences that alternate sign with increasing parameters \((m, n, ...\))

\[
m(a_{m+1} - a_{m-1}) + 2a_m = 0 \quad [m > 0]
\]
Bianchi I LRS constraint:

\[ 2d_2(m)[t_{m+1,n+1} - t_{m+1,n-1} - t_{m-1,n+1} + t_{m-1,n-1}] \]
\[ +d(n)[t_{m+2,n} - 2t_{m,n} + t_{m-2,n}] = 0 \]

To simplify this recursion relation, we notice that it is separable into two equations:

\[ a_{m+1} - a_{m-1} = (2\lambda/m)a_m \]
\[ b_{n+1} - b_{n-1} = -\lambda d(n)b_n \]

Because \( d(n) \simeq 1/2m \), these are essentially the same equation (up to scaling of the separation parameter \( \lambda \)).

We insist on "pre-classicality" – wave functions are smooth at large values of the parameters; for example, in a one-parameter sequence \( a_m \),

\[ (a_m - a_{m-1}) \to 0 \quad m \to \infty \]
Generating function techniques

Instead of dealing with a sequence \( \{a_m\} \), write these as coefficients of a generating function \( F(x) \)

\[
\{a_m\} \iff F(x) = \sum_{m=0}^{\infty} a_m x^m
\]

Operations on \( \{a_m\} \) map to operations on \( F(x) \):

\[
\{a_{m+1}\} \iff \frac{F(x) - a_0}{x}
\]

\[
\{ma_m\} \iff x \frac{dF(x)}{dx}
\]

The generalization for an arbitrary number of parameters \( m_i \) is obvious...

We write the Hamiltonian constraint for wave functions as a differential equation for the generating functions \( F(x_i) \) of our sequence
In general, when you solve the PDE for the generating function \( F(x_i) \), it has singularities at various points

- Some singularities are "bad", such as \((1 + x)^{-1}\), where the coefficients \( \{a_m\} \) of the series expansion oscillate:
  \[
  \frac{1}{1 + x} = 1 - x + x^2 - x^3 + \cdots
  \]

- Others are "good", such as \((1 - x)^{-1}\), where the coefficients \( \{a_m\} \) are constant
  \[
  \frac{1}{1 - x} = 1 + x + x^2 + x^3 + \cdots
  \]

Note this has to be a simple pole to avoid coefficients increasing without limit, since
\[
\frac{1}{(1 - x)^2} = 1 + 2x + 3x^2 + 4x^3 + \cdots
\]

Pre-classicality, asymptotic behavior of sequences \( \Leftrightarrow \) conditions on poles of generating functions
Solving the recursion relation

\[ a_{m+1} - a_{m-1} = (2\lambda/m)a_m \]

turns out to be equivalent (for \( m > 0 \)) to solving the ODE

\[ \frac{d}{dx}[(1-x^2)G(x)] - 2\lambda G(x) = a_0 \]

where

\[ F(x) = a_0 + xG(x) \]

Because of the \((1-x^2)\) term multiplying \( dG(x)/dx \), the solution \( G(x) \) can have poles at \( x = \pm 1 \).
The generating function for the relation with \( \lambda = -1 \) will have a double pole at \( x = -1 \); requiring

\[ [(1 + x)^2 F(x)]_{x=-1} = 0 \]

gives a condition on the ratio \( a_1/a_0 \) of the initial data of the sequence.
The advantage of using generating functions is that we can obtain the values $a_m$ for any $m$, not just integers.

For example, with $a_0 = 0$ and $\lambda > 0$, we have that the generating function of the sequence is

$$F(x) = a_1 x (1 + x)^{\lambda - 1} (1 - x)^{-\lambda - 1}$$

The coefficients of the Taylor series give the values $a_m$:

$$a_m = a_1 \sum_{j=0}^{m-1} (-1)^j \binom{\lambda - 1}{m - j - 1} \binom{-\lambda - 1}{j}$$

By using hypergeometric functions, this can be defined for all real values of $m$. 
Plot of $a_m$ when $\lambda = 3/2$

(using $1/m$ in the recursion relation)
When $d(n)$ is used in the recursion relation, sequences must be numerically evolved from large $m$; again choosing $\lambda = 3/2$...

Each line on the LHS represents a non-integer $b$, where $m = n + b$, ($n$ an integer).
The scaling of the sequence varies as a function of the non-integer "remainder" $b$...

Plot of the ratio $a_M/a_{-M}$ for large $M$ versus $b$ (for $\lambda = 3/2$)
For $\lambda < 0$, solutions are more limited – either there are no pre-classical solutions (for $a_0 = 0$) or the sequence is pre-classical only on one side ($a_0 \neq 0$).

Plot with $\lambda = -1, m = n + 3/4$
Classical metric given by

\[ ds^2 = -dt^2 + \sum_{i=1}^{3} a_i^2(t) dx_i^2 \]

for scale factors \( a_i(t) \)

Hamiltonian constraint for vacuum case in LQC:

\[
d(m_1)[t_{m_1,m_2+1,m_3+1}-t_{m_1,m_2-1,m_3+1}-t_{m_1,m_2+1,m_3-1}+t_{m_1,m_2-1,m_3-1}] \\
+ \text{(cyclic)} = 0
\]

This can be written as a product of three separable sequences, satisfying the relations discussed previously, with

\[ \lambda_1 \lambda_2 + \lambda_1 \lambda_3 + \lambda_2 \lambda_3 = 0 \]

Since pre-classicality requires \( \lambda_k > 0 \), there are no pre-classical solutions at all for this recursion relation!
Two of the three scale factors $a_i(t)$ in the metric on the previous slide are the same.

Slope goes to zero for $m \gg 1$; essentially no variation in $n$ direction.
Using numerical techniques adapted to solving these difference equations (Connors and Khanna, gr-qc/0509081), one can find solutions when a cosmological constant is added:

For large $m, n$, these numerical solutions should be matchable to the semi-classical solution of the WdW equation.
The recursion relation away from the horizon ($m \geq 2$) is given by (Ashtekar and Bojowald, gr-qc/0509075)

\[
(\sqrt{|n + 1|} + \sqrt{|n|})(s_{m+1,n+1} - s_{m-1,n+1}) \\
+ (\sqrt{|n - 1|} + \sqrt{|n|})(s_{m-1,n-1} - s_{m+1,n-1}) \\
+ (\sqrt{|n + 1/2|} - \sqrt{|n - 1/2|})[(n + 1)s_{m+2,n} \\
- \kappa ns_{m,n} + (n - 1)s_{m-2,n}] = 0
\]

The constant $\kappa$ depends on the Barbero-Immirzi parameter $\gamma$ and a quantum ambiguity $\delta$

Again, this is separable into two relations, which are more complicated than those for Bianchi I
For the "time" sequence $b_n$ which passes through the singularity $n = 0$, there is no restriction on the value of the separation parameter $\lambda$...

When $\lambda < -2$, initial data is free of restrictions and the solution grows without bound:
When $\lambda > -2$, the ratio $b_1/b_0$ is fixed to ensure no oscillations far from the origin.
For the "spatial" $a_m$ sequences, the generating function has the polynomial $(x^4 - \kappa x^2 + 1)$ in front of the derivative $dG/dx$. Since $\kappa \geq 0$, this means there are poles at $x = \pm x_0, \pm 1/x_0$.

One of these must be avoided to prevent oscillations at large $m$; another will cause the sequence to grow without limit (choose $a_m \to 0$ as $m \to \infty$ as boundary condition).

Since there are four initial values $(a_0, \ldots, a_3)$, with at most two to fix, it seems that we again have freedom in our choices for all $\lambda$
Conclusions, future work

• Generating function methods are useful in finding the space of pre-classical solutions for the discrete Hamiltonian constraint in LQC

• Against expectations, pre-classical solutions for vacuum anisotropic models studied so far are non-existent (Bianchi I) or very limited (Bianchi I LRS)

• It appears there is sufficient freedom in the Schwarzschild interior to build up generic wave forms at the horizon of the black hole

• Further avenues to explore:
  – addition of matter (e.g. scalar field)
  – physical wave functions via group averaging