# Finiteness and Positivity for the Lorentzian Barrett-Crane partition function

Dan Christensen and Wade Cherrington
University of Western Ontario

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Outline:

Face factoring

Finiteness

Positivity

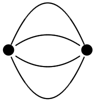
# The usual formulation of the Lorentzian BC model

Let $\Delta$ be a triangulation of a closed 4-manifold. $F$ = dual faces = triangles, $E$ = dual edges = tets, $V$ = dual vertices = 4-simplices.

Partition function for $\Delta$:

$$\mathcal{A}(\Delta) \equiv \underbrace{\int_0^\infty \cdots \int_0^\infty}_{f \in F} \left( \prod_{f \in F} \mathcal{A}_f \right) \left( \prod_{e \in E} \mathcal{A}_e \right) \left( \prod_{v \in V} \mathcal{A}_v \right) \prod_{f \in F} p_f^2 \, dp_f \, ,$$

where $$\mathcal{A}_v = 10j \text{ symbol} \, ,$$

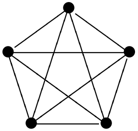
$\mathcal{A}_e$ = eye diagram =  = $\Theta_4$ ,

and $$\mathcal{A}_f = 1 \, .$$

(This is the Perez-Rovelli normalization.)

# $\mathcal{A}_v$

$\mathcal{A}_v$ is the $10j$ symbol



$$\mathcal{A}_v = \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \prod_{f \ni v} K_{p_f}(\phi_v^f) \prod_{\substack{e \ni v \\ \text{one skipped}}} dx_e \,,$$

where

$$K_{p_f}(\phi_v^f) = \frac{\sin(p_f\, \phi_v^f)}{p_f \sinh(\phi_v^f)}$$

and $\phi_v^f$ is the hyperbolic distance between $x_e$ and $x_{e'}$, where $e$ and $e'$ are the two tets in $v$ that are separated by the triangle $f$.

# $\mathcal{A}_e$

$$\mathcal{A}_e = \Theta_4(p_1, \ldots, p_4) = \frac{2}{\pi}\int_0^\infty \frac{\sin(p_1 r)\sin(p_2 r)\sin(p_3 r)\sin(p_4 r)}{p_1 p_2 p_3 p_4 \sinh^2(r)}\, dr,$$

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This can be integrated exactly, but we will see it is better to leave it unintegrated.

# Summary of usual formulation

$$\mathcal{A}(\Delta) \equiv \underbrace{\int_0^\infty \cdots \int_0^\infty}_{f\in F} \left(\prod_{e\in E} \mathcal{A}_e\right)\left(\prod_{v\in V} \mathcal{A}_v\right) \prod_{f\in F} p_f^2 \, dp_f \,,$$

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Problem: the $10j$ symbols ($\mathcal{A}_v$) are very hard to compute!

# Face factoring (Cherrington, gr-qc/0508088)

Similar to dual variables idea of Pfeiffer (CQG 2002). Exchange the order of integration, to obtain:

$$\mathcal{A}(\Delta) = \left( \overline{\prod_e} \int_0^\infty dr_e \right) \left( \overline{\prod_v} \left( \overline{\prod_{e \ni v, e \neq e_0^v}} \int_{\mathbb{H}^3} dx_e^v \right) \right) \mathcal{A}(\Delta, x_e^i)$$

where $$\mathcal{A}(\Delta, x_e^i) \equiv \frac{1}{\text{lots of sinh's}} \prod_f \int_0^\infty F_f(p_f, \phi_v^f, r_e)\, dp_f$$

is expressed using the face factors

$$F_f(p_f, \phi_v^f, r_e) \equiv \frac{\sin(p_f\, \phi_1^f) \cdots \sin(p_f\, \phi_m^f) \sin(p_f\, r_{e_1^f}) \cdots \sin(p_f\, r_{e_m^f})}{p_f^{2m-2}}\,.$$

Here $m$ is the number of vertices = number of edges in the face $f$.

# Amazing facts

At first glance, the exchange of integration looks like a dumb idea: we've traded a high-dimensional integral over the $p_f$'s for an even higher-dimensional integral over the $r_e$'s and the $x_e^v$'s. But:

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- The integrated face factors are non-negative.

All of these have implications for numerical computations; see Wade Cherrington's poster at this conference.

This talk will focus on some of the theoretical implications.

# Finiteness

Consider the following analog of the 10$j$ symbol:

$$\mathcal{A}'_v = \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \int_{\mathbb{H}^3} \prod_{0 \le i < j \le 4} k(\phi_{ij})\, dx_1 \cdots dx_4 \,,$$

where

$$k(\phi) = \frac{\phi^\alpha}{\sinh(\phi)}\,, \quad \alpha \ge 0\,,$$

and $\phi_{ij}$ is the hyperbolic distance from $x_i$ to $x_j$.

For $\alpha < 1$, this kernel is divergent at $\phi = 0$, and the product of ten of these kernels has a complicated singularity structure.

Cherrington shows that if $\mathcal{A}'_v$ is convergent for each $\alpha \ge 0$, then the face-factored Lorentzian partition function is absolutely convergent, even for degenerate triangulations. This was previously shown by Crane, Perez and Rovelli for non-degenerate triangulations.

I've proved that these singular $10j$ symbols are absolutely convergent. First, a simple observation:

**Lemma**

*Let $y$ be a point in $\mathbb{H}^3$. Then the integral*

$$\int_{\mathbb{H}^3} (k(\phi(x,y)))^m \, dx = \int_{\mathbb{H}^3} \left( \frac{\phi(x,y)^\alpha}{\sinh(\phi(x,y))} \right)^m \, dx$$

*is absolutely convergent for $2 < m < 3$ and any $\alpha \geq 0$. Moreover, the value is independent of $y$.*

Proof: When $2 < m$, it is integrable at infinity. And when $m < 3$, it is integrable at the origin.
It is clearly translation invariant.

# Sketch of proof of finiteness of singular $10j$:

Write $k_{ij} = k_{ji} = k(\phi_{ij})$. The integrand is

$$(k_{01} k_{14} k_{43} k_{32} k_{20})(k_{04} k_{42} k_{21} k_{13} k_{30})$$

So we get the following upper bound on the integrand:

$$\begin{aligned}(k_{01} k_{14} k_{43} k_{32} k_{20})&(k_{04} k_{42} k_{21} k_{13} k_{30}) \\ &\leq \frac{1}{2}\left((k_{01} k_{14} k_{43} k_{32} k_{20})^2 + (k_{04} k_{42} k_{21} k_{13} k_{30})^2\right).\end{aligned}$$

(All of the inequalities discussed here hold pointwise, for each $x_0, \ldots, x_4$ in $\mathbb{H}^3$.)

The two terms on the right-hand side are symmetrical, so it suffices to show that the first is integrable.

We will show that

$$(k_{01}k_{14}k_{43}k_{32}k_{20})^2$$

is integrable. Using the convexity of exp, we have

$$\begin{aligned}(k_{01}k_{14}k_{43}k_{32}k_{20})^2 &= (k_{01}^2 k_{14}^2 k_{43}^2 k_{32}^2 k_{20}^2) \\ &\le \frac{1}{5}\left((k_{14}^2 k_{43}^2 k_{32}^2 k_{20}^2)^{5/4} + \cdots + (k_{01}^2 k_{14}^2 k_{43}^2 k_{32}^2)^{5/4}\right) \\ &= \frac{1}{5}\left((k_{14}k_{43}k_{32}k_{20})^{5/2} + \cdots + (k_{01}k_{14}k_{43}k_{32})^{5/2}\right).\end{aligned}$$

We'll focus on first term; rest similar.

First term is $(k_{14}k_{43}k_{32}k_{20})^{5/2}$. Order the integrations as follows:

$$\int_{\mathbb{H}^3}\int_{\mathbb{H}^3}\int_{\mathbb{H}^3}\int_{\mathbb{H}^3}(k_{14}k_{43}k_{32}k_{20})^{5/2}\,dx_1\,dx_4\,dx_3\,dx_2.$$

The innermost integral is

$$\int_{\mathbb{H}^3} k_{14}^{5/2}\,dx_1,$$

which is finite and independent of $x_4$ by the Lemma. Similarly, the next integral is

$$\int_{\mathbb{H}^3} k_{43}^{5/2}\,dx_4,$$

which produces a constant. And so on. □

The same argument gives a simple proof that the usual Lorentzian $10j$ symbol is finite (known by work of Baez and Barrett). Note that no hyperbolic geometry or $\epsilon$'s were needed!

It also proves that the various causal $10j$ symbols introduced by Livine-Oriti and by Pfeiffer are finite, in both the Lorentzian and Riemannian cases. The kernels are singular in these cases, and these are new results.

In fact, the method works for any kernel $K$ such that

$$\int_{\mathbb{H}^3} K(\phi(x,y))^{5/2}\,dx$$

is absolutely convergent.

It extends to other spin networks as well, but so far not to the $6j$ symbol.

# Positivity (Cherrington-Christensen, gr-qc/0509080)

Recall that we rewrote the partition function as

$$\mathcal{A}(\Delta) = \left( \overline{\prod_e} \int_0^\infty dr_e \right) \left( \overline{\prod_v} \left( \overline{\prod_{e \ni v, e \neq e_0^v}} \int_{\mathbb{H}^3} dx_e^v \right) \right) \mathcal{A}(\Delta, x_e^i)$$

where

$$\mathcal{A}(\Delta, x_e^i) \equiv \frac{1}{\text{lots of sinh's}} \prod_f \int_0^\infty F_f(p_f, \phi_v^f, r_e)\, dp_f$$

and

$$F_f(p_f, \phi_v^f, r_e) \equiv \frac{\sin(p_f\, \phi_1^f) \cdots \sin(p_f\, \phi_m^f) \sin(p_f\, r_{e_1^f}) \cdots \sin(p_f\, r_{e_m^f})}{p_f^{2m-2}} .$$

So if we can show that

$$\int_0^\infty \frac{\sin(a_1 t)\cdots\sin(a_n t)}{t^{n-2}}\,dt \qquad\qquad \begin{array}{l} t = p_f \\ a_i = \phi_v^f \text{ or } r_e \end{array}$$

is non-negative for all $a_1, \ldots, a_n > 0$, this will prove that the Lorentzian partition function is non-negative.

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**Theorem**

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### Theorem

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For $n = 3$, this can be checked by doing the integral symbolically.
For $n \geq 4$, we can use Fourier methods:
Since the integrand is even, it suffices to show that

$$\int_{-\infty}^{\infty} t^2 \prod_{i=1}^{n} a_i \operatorname{sinc}(a_i t)\,dt$$

is non-negative, where $\operatorname{sinc}(t) = sin(t)/t$.

Use that

$$\int_{-\infty}^{\infty} t^2 \prod_{i=1}^{n} a_i \,\mathrm{sinc}(a_i t)\, dt = \mathcal{F}\left( t^2 \prod_{i=1}^{n} a_i \,\mathrm{sinc}(a_i t) \right)(k = 0)\,.$$

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Computing the Fourier transform gives

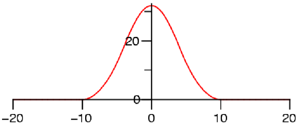
$$\begin{aligned}
\mathcal{F}\left( t^2 \prod_{i=1}^{n} a_i \operatorname{sinc}(a_i t) \right)(k) &= i\frac{d}{dk}\mathcal{F}\left( t \prod_{i=1}^{n} a_i \operatorname{sinc}(a_i t) \right)(k) \\
&= -\frac{d^2}{dk^2}\mathcal{F}\left( \prod_{i=1}^{n} a_i \operatorname{sinc}(a_i t) \right)(k) \\
&= -\frac{\pi}{2^{n-1}}\frac{d^2}{dk^2} \mathop{*}_{\mathrm{i}=1}^{\mathrm{n}} \chi_{a_i}(k)\,,
\end{aligned}$$

where

$$\chi_{a_i}(k) = \begin{cases} 1, & \text{for } -a_i < k < a_i \\ 0, & \text{otherwise} \end{cases} \quad .$$

Now use that the convolution

$$\mathop{*}_{i=1}^{n} \chi_{a_i}(k)$$




is an even bump, i.e., it is symmetrical about 0 and decreases as you go away from zero. This implies that

$$\frac{d^2}{dk^2} \mathop{*}_{i=1}^{n} \chi_{a_i}(k = 0)$$

is non-positive, giving the desired result.

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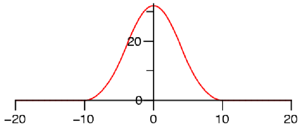


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- Some delicate issues have been skipped; one has to be careful when evaluating a Fourier transform at a point.
- Similar methods can handle models with different normalizations, although the analysis gets much more technical.

# Conclusions

The Lorentzian Barrett-Crane model is in many ways easier to understand in the face-factored or dual variables form:

- It can be shown to be finite with a simple argument.
- It can be shown to be non-negative with a simple argument.
- Expectation values of observables can be computed using statistical methods, even though we don't know that the "paths" in the usual formulation have non-negative amplitudes. (We conjecture that they do.)

This has opened the door for numerical work on this model. See Cherrington's poster for more on this.

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