Semiclassical States and

Constrained Systems

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What to do to find semiclassical states?

As has been repeatadely said in this conference:

- The semiclassical limit of LQG is very important.
- The semiclassical limit of LQG is very difficult.
- In the semiclassical limit of LQG *dynamics* is very important.

But just *how* important it is, do we really know?

One of the motivations for studying this problem is to gain intuition on the importance of dynamics in the definition of semiclassical states.

Another motivation is even more basic:

What is a semiclassical state?

We will give partial answers to these question, to pave the way for LQG.

What are semiclassical States?

Fix a point α in a linear phase space Γ with coordinates (q_i^0, p_i^0). Our task is to spell out what we mean by a semi-classical quantum state which is 'peaked at this classical state'. One generally has the following idea in mind:

- A semi-classical state Φ_α should be such that, for all well-behaved functions $F(q_i, p_i)$ on phase space, the expectation values $\langle \Phi_\alpha | \hat{F} | \Phi_\alpha \rangle$ are close to $F(q_i^0, p_i^0)$ and the fluctuations small.

However, such semi-classical states simply don't exist unless the class of observables is greatly restricted.

Take a harmonic oscillator. A coherent states Ψ_α represents semi-classical states peaked at the point α of the phase space Γ.

For, q, p, H, where H is the Hamiltonian, Ψ_α would satisfy the above criteria (if the words 'close to' and 'small' are interpreted appropriately; as we will see below).

However, if the set also includes $\mathrm{e}^{H/\epsilon}$ with $\epsilon \leq \hbar\omega$, coherent states would strongly violate the criteria. Of course, for the harmonic oscillator the new observable is rather strange and it is difficult to justify its inclusion in the list on physical grounds, but there are (quantum gravity) systems for which this observable is important.
Thus, there is no canonical notion of semi-classicality for the system, independent of one's choice of observables.

The first lesson is that to ask for semi-classical states, one must first specify a class of observables for which the states are to be semi-classical.

The second subtlety has to do with the notion of fluctuations.

The requirement that the fluctuations of an observable $\hat{F}$ in a state Ψ be small is generally formulated as:

$$\frac{(\Delta\hat{F})^2_\Psi}{|\langle\Psi|\hat{F}|\Psi\rangle|^2} \equiv \frac{\langle\Psi|\hat{F}^2|\Psi\rangle - [\langle\Psi|\hat{F}|\Psi\rangle]^2}{|\langle\Psi|\hat{F}|\Psi\rangle|^2} < \delta^2 \; , \tag{1}$$

where δ denotes the 'tolerance' one wishes to allow.

There is, however, the following problem with this proposal: if the expectation value $\langle\hat{F}\rangle_\Psi$ vanishes, the requirement can never be met!

For constrained systems we would like to consider kinematical coherent states which are peaked at a point on the constraint surface with only a small spread and compare them with physical semi-classical states. Eq. (1) forbids us from taking such states.

Considerations regarding experimental limitations on our measurements lead us to a specific notion of semi-classicality that will be used here.

Given a state, if one were allowed to make arbitrarily accurate measurements of any observable, one would find deviations from the classical behavior.

Thus, a quantum state can be well approximated by a classical one only if the experimental accuracy is limited. To test semi-classicality, we must supply information about these experimental limitations, i.e., tolerances which are fixed at the outset. We will need two sets of numbers, one specifying the tolerance in the accuracy of the expectation value, and the other one that in fluctuations.

A state Ψ_α will be said to be peaked at the point $\alpha \in \Gamma$ and semi-classical with respect to a given set of observables F_i if

$$|\langle\Psi_\alpha|\hat{F}_i|\Psi_\alpha\rangle - F_i(\alpha)| < \epsilon_i \quad \text{and} \quad (\Delta\hat{F}_i)_{\Psi_\alpha} < \delta_i \ , \tag{2}$$

where ϵ_i and δ_i are pre-specified tolerances determined by the desired experimental accuracy.

Strategy

The idea is to use the group averaging technique to extract physical semi-classical states $\Psi_\alpha^{\rm phy}$ —i.e., semi-classical states which are annihilated by the constraint operator— starting from standard coherent states Ψ_α in $\mathcal{H}_{\rm kin}$.

Since the notion of semi-classicality is relative to a set of observables, we will begin by fixing the set of Dirac observables $\mathcal{O}_i$ of interest, together with tolerances ϵ_i and δ_i. By definition, the physical states $\Psi_\alpha^{\rm phy}$ will be semi-classical if they satisfy (2).

The issue then is:

Can we make a suitable choice of Ψ_α that will guarantee that the $\Psi_\alpha^{\rm phy}$ are semi-classical?

An example of a sufficient condition for the answer to be affirmative is:

$$|\langle \hat{\mathcal{O}}_i \rangle_{\rm phy} - \langle \Psi_\alpha | \hat{\mathcal{O}}_i | \Psi_\alpha \rangle| < \tfrac{1}{2}\,\epsilon_i \quad \text{and}$$

$$|(\Delta \hat{\mathcal{O}}_i)_{\Psi_a^{\rm phy}} - (\Delta \hat{\mathcal{O}}_i)_{\Psi_a}| < \tfrac{1}{2}\,\delta_i \ .$$

Recall that, given any $\Psi \in \mathcal{H}_{\rm kin}$, the group average,

$$\Psi_{\rm phy} := \frac{1}{\Lambda} \int_0^\Lambda {\rm d}\lambda \, {\rm e}^{-{\rm i}\lambda \hat{C}} \, \Psi \ , \qquad (3)$$

where Λ is chosen such that ${\rm e}^{-{\rm i}\Lambda\hat{C}} = 1$ Then $\Psi_{\rm phy}$ satisfies the constraint, $\hat{C} \cdot \Psi_{\rm phy} = 0$, and is thus a physical state.

The theories we will consider will involve:

1. Linear and quadratic constraints and
2. General linear and quadratic observables.

Linear and Quadratic Constraints

Let us now consider constraints of the type

$$C := K_i q_i + \tilde{K}_i p_i - \Delta = 0 \ , \tag{4}$$

where $K_i, \tilde{K}_i$ and Δ are any real constants.

The constraint (4) can be written as $C := \bar{\kappa}_i z_i + \kappa_i \bar{z}_i - \Delta = 0$, where the complex numbers κ_i are related to K_i, $\tilde{K}_i$ in the obvious manner. Given any coherent state Ψ_α in the kinematical Hilbert space, where α is not necessarily on the constraint surface, it is easy to verify that

$$\hat{U}(\lambda)\Psi_\alpha := \mathrm{e}^{-\mathrm{i}\lambda\hat{C}}\,\Psi_\alpha = \mathrm{e}^{-\mathrm{i}\lambda\, C(\alpha)}\,\Psi_{\alpha(\lambda)} \ , \tag{5}$$

where $C(\alpha)$ is the value of the classical constraint C at α and $\alpha_j(\lambda) = \alpha_j - \mathrm{i}\lambda\kappa_j$ with α_j labelling the initial phase space point α given by,

$$\alpha_i := \frac{q_i^0}{\sqrt{2}\,\ell_i} + \mathrm{i}\,\frac{\ell_i\, p_i^0}{\sqrt{2}\,\hbar} \, . \tag{6}$$

we can always orient our basis so that we have $K_i q_i + \tilde{K}_i p_i = q_1$, and the constraint reduces then to the simple form

$$C = q_1 - \Delta = 0\ . \tag{7}$$

For simplicity, let us choose all ℓ_i to be equal. Then in the q-representation, the action of $\hat{U}(\lambda)$ on Ψ_α further simplifies

$$\hat{U}(\lambda)\Psi_\alpha(q) = \mathcal{N}\ \mathrm{e}^{\mathrm{i}\lambda\Delta}\ \mathrm{e}^{\mathrm{i}(-\lambda q_1 + p^0\cdot\, q/\hbar)}\ \mathrm{e}^{-|q-q^0|^2/2\ell^2}\ , \tag{8}$$

The physical state can also be readily calculated:

$$\Psi_\alpha^{\mathrm{phy}}(q) := \int \mathrm{d}\lambda\, \hat{U}(\lambda)\Psi_\alpha(q)\ =$$

$$= 2\pi\mathcal{N}\,\delta(q_1 - \Delta)\,\mathrm{e}^{\mathrm{i}(p^0\cdot\, q)/\hbar}\,\mathrm{e}^{-|q-q^0|^2/2\ell^2}\ .$$

We consider general polynomials $F(\bar{z}_I, \bar{z}_J)$ and their *normal ordered* quantum versions,

$$\hat{F} = :F(\alpha_I^\dagger, a_J):$$

. The expectation values of these operators in the kinematic coherent states Ψ_α are just the values $F(\bar{\alpha}_I, \alpha_J)$ of the classical functions F, evaluated at the points α of the phase space:

$$\langle \Psi_\alpha | \hat{F} \, | \Psi_\alpha \rangle = F(\bar{\alpha}_I, \alpha_J) \ . \tag{9}$$

the fluctuations are given by

$$(\Delta \hat{F})^2_\alpha = \langle \hat{F}^2 \rangle_\alpha - (\langle \hat{F} \rangle_\alpha)^2 = G(\bar{\alpha}_I, \alpha_J) - (F(\bar{\alpha}_I, \alpha_J))^2 \, . \tag{10}$$

Since $\hat{F}$ and $\hat{G}$ do not involve $\hat{a}_1$ and $\hat{a}_1^\dagger$, it is easy to calculate the expectation values and fluctuations also in the (normalized) physical states $(\ell^2/4\pi)^{1/4} \, \Psi_\alpha^{\rm phy}(x)$. One obtains that

$$\langle \hat{F} \rangle_\alpha^{\rm phy} \ = \ F(\alpha_I, \bar{\alpha}_I),$$

and

$$(\Delta \hat{F}_\alpha^{\rm phy})^2 = G(\alpha_I, \bar{\alpha}_I) - (F(\alpha_I, \bar{\alpha}_I))^2 \, .$$

If Ψ_α is semi-classical, then $\Psi_\alpha^{\rm phy}$ will also satisfy our semi-classicality criteria.

Example: Gauss constraint in Maxwell

$$A_a(x) \;=\; \frac{1}{(2\pi)^{3/2}} \;\int \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}k\cdot x} \,\Big(q_1(k)\,\hat{k}_a + q_2(k)\,m_a + q_3(k)\,\bar{m}_a\Big)$$

$$E^a(x) \;=\; \frac{-1}{(2\pi)^{3/2}} \;\int \mathrm{d}^3k\, \mathrm{e}^{\mathrm{i}k\cdot x} \,\Big(p_1(k)\,\hat{k}^a + p_2(k)\,m^a + p_3(k)\,\bar{m}^a\Big)$$

The phase space can be coordinatized by pairs $(q_i(k), p_i(k))$. The fields $q_i(k)$, $p_j(k)$ are canonically conjugate in the sense that

$$\{q_i(-k), p_j(k')\} = \delta_{ij}\,\delta^3(k, k')\ . \tag{11}$$

The standard Kähler structure is given by the positive and negative frequency decomposition. The holomorphic coordinates $z_i(k)$ are now given by

$$z_j(k) = \frac{1}{\sqrt{2}}\,(\sqrt{|k|}\,q_j(k) - \frac{\mathrm{i}}{\hbar\sqrt{|k|}}\,p_j(k))\ . \tag{12}$$

The Gauss law $D_a E^a = 0$ is equivalent to $p_1(k) = 0$ which, in turn, can be recast as an infinite set of commuting constraints,

$$C_f(k) := \int \mathrm{d}^3k\, \bar{f}(k)(z_1(k) - \bar{z}_1(k)) = 0\ , \tag{13}$$

one for each regular function $f(k)$ in the momentum space (e.g., an element of the Schwartz space in R^3). $\mathcal{H}_{\rm kin}$ is the Fock space obtained by operating repeatedly with the creation operators on the vacuum state $|0\rangle$. For α, a coherent state Ψ_α, peaked at $z_i(k) = \alpha_i(k)$, can now be constructed in $\mathcal{H}_{\rm kin}$:

$$|\Psi_\alpha\rangle = \mathrm{e}^{\int (\mathrm{d}^3k/|k|)\,(\alpha(k)\cdot \hat{a}^\dagger(k) - \bar{\alpha}(k)\cdot\hat{a}(k))}\,|0\rangle\ . \qquad (14)$$

Assume $\alpha_i(k)$ lies on the constraint surface, i.e., that $\alpha_1(k)$ is real.

By group average procedure, the action of the distribution $(\Psi^{\rm phy}_\alpha|$ on an *arbitrary* Ψ_β is given by

$$(\Psi^{\rm phy}_\alpha|\Psi_\beta\rangle = \langle\Psi_{\hat{\alpha}}|\Psi_{\hat{\beta}}\rangle\ , \qquad (15)$$

$\hat{\alpha}_1(k) = 0,\ \hat{\alpha}_I(k) = \alpha_I(k)$ and $\hat{\beta}_1(k) = 0,\ \hat{\beta}_I(k) = \beta_I(k)$.

Effectively 'removing from Ψ_α all information about the longitudinal modes, keeping the transverse modes intact'.

By the general considerations, expectation values and fluctuations of Dirac observables (independent of longitudinal modes) coincide with those of the kinematical coherent states.

Quadratic Constraints

$C(q_i, p_i) := S_{ij}\, q_i q_j + \Lambda\, S_{ij}\, p_i p_j + A_{ij}\, q_i p_j - \Delta = 0$,

S_{ij} is a symmetric matrix, A_{ij} an anti-symmetric matrix, Λ a constant with dimensions $[L^2/(\text{Action})]^2$, and Δ a real constant. For example,

$$C_{\vec{N}}(q,p) \;=\; \int_M P^{ab}(x)\, \pounds_{\vec{N}}\, q_{ab}(x)\, \mathrm{d}^3x \;, \qquad (16)$$

is in this class. In terms of the holomorphic coordinates z_i of Eq (**??**), the constraints we consider can be written as

$$C(q_i, p_i) := \kappa_{ij} z_i \bar{z}_j - \Delta = 0 \;, \qquad (17)$$

Then, using as before normal ordering, the quantum constraint operator becomes

$$\hat{C} = \kappa_j\, \hat{N}_j - \Delta\, \hat{1} \;, \qquad (18)$$

with $\hat{N}_j$ the jth number operator, $\hat{N}_j = \hat{a}_j^\dagger \hat{a}_j$ (where there is no summation over j)

$$\mathrm{e}^{-\mathrm{i}\lambda\hat{C}}\, |\Psi_\alpha\rangle = \mathrm{e}^{\mathrm{i}\lambda\Delta}\, |\Psi_{\alpha(\lambda)}\rangle \;,$$

A linear observable is one of the form:

$$\mathcal{O} = \bar{F}_i z_i + F_i \bar{z}_i$$

We get that for the expectation values,

$$\langle \hat{\mathcal{O}} \rangle_\alpha^{\rm phy} = \frac{\langle \Psi_\alpha^{\rm phy} | \, \hat{\mathcal{O}} \, | \Psi_\alpha^{\rm phy} \rangle}{\| \Psi_\alpha^{\rm phy} \|^2} = \mathcal{O}(\alpha) \; , \tag{19}$$

the same as the classical value. Similarly, for the fluctuation we calculate

$$(\Delta \hat{\mathcal{O}})^2_{\rm phy} = (\Delta \hat{\mathcal{O}})^2_{\rm kin} = F_i \bar{F}_i \; . \tag{20}$$

That is, the fluctuations are preserved.

What about quadratic Observables?

Well, for general quadratic observables, the exact expressions for expectation values and fluctuations look kind of messy.

So, we illustrate with some examples:

Example 2: Constrained total energy

This example is very popular, and looks like,

$$\tilde{C} := \Sigma_i \frac{p_i^2}{2m} + kq_i^2 - \tilde{\Delta} = 0 \; . \tag{21}$$

can be re-expressed as

$$C := \frac{1}{\hbar\omega}\, \tilde{C} = z_1\bar{z}_1 + z_2\bar{z}_2 - \Delta = 0 \; , \tag{22}$$

where $\Delta = \tilde{\Delta}/\hbar\omega$. The kinematic phase space Γ is R^4; the constraint surface $\bar{\Gamma}$ is a 3-sphere; and the gauge orbits generated by the constraint function C provide a Hopf fibration of $\bar{\Gamma}$.

We need a set of at least three Dirac observables to separate points of $\hat{\Gamma}$. A convenient choice is

$$L_1 = \mathrm{Re}\; z_1\bar{z}_2, \quad L_2 = \mathrm{Im}\; z_1\bar{z}_2, \quad L_3 = \tfrac{1}{2}\,(z_1\bar{z}_1 - z_2\bar{z}_2) \; .$$

Our constraint operator is

$$\hat{C} = \hat{a}_i\hat{a}_i^\dagger - \Delta = \Sigma_i \hat{N}_i - \Delta \; . \tag{23}$$

for $i = 1, 2$.

The physical coherent states are:

$$|\Psi_\alpha^{\mathrm{phy}}\rangle = \frac{\mathrm{e}^{-(|\alpha_1|^2+|\alpha_2|^2)/2}}{2\pi} \int_0^{2\pi} \mathrm{d}\lambda \sum_{n,m=0}^{\infty} \frac{\alpha_1^n\,\alpha_2^m}{\sqrt{n!}\,\sqrt{m!}}\,\mathrm{e}^{-\mathrm{i}\lambda\hat{C}}\,|n,m\rangle\ . \tag{24}$$

The Fock basis $|n,m\rangle$ satisfies $\hat{C}|n,m\rangle = n+m-\Delta$, where $\Delta = k := n+m$ is an integer, then,

$$|\Psi_\alpha^{\mathrm{phy}}\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\lambda\,\mathrm{e}^{\mathrm{i}\lambda\Delta}\ |(\alpha_1\mathrm{e}^{-\mathrm{i}\lambda}),(\alpha_2\mathrm{e}^{-\mathrm{i}\lambda})\rangle\ . \tag{25}$$

The physical Hilbert space is finite-dimensional (due to the compactness of the reduced phase space), with $\dim(\mathcal{H}_{\mathrm{phy}}) = k+1$, and the projection operator that projects the kinematical coherent state to the subspace spanned by kets of the form $|n,k-n\rangle$ for a fixed value of k

Let us begin with the term $a_1^\dagger a_1$, and compute its expectation value $\langle\Psi_\alpha|\,a_1^\dagger a_1|\Psi_\alpha\rangle_{\mathrm{phy}}$.

In this case we have

$$
\begin{aligned}
\langle\Psi_\alpha|\, a_1^\dagger a_1|\Psi_\alpha\rangle_{\rm phy} \;&=\; |\alpha_1|^2\, \frac{\mathrm{e}^{-\tilde{E}}}{2\pi\mathrm{i}} \oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta}\, \zeta^{-(\tilde{E}-1)}\, \mathrm{e}^{\tilde{E}\,\zeta} \\
&=\; |\alpha_1|^2\, \frac{\mathrm{e}^{-k}}{(k-1)!}\, k^{k-1}\,.
\end{aligned} \tag{26}
$$

The norm of the physical states $\Psi_\alpha^{\rm phy}$, is given by

$$
\begin{aligned}
\|\Psi_\alpha^{\rm phy}\|^2 \;&=\; \frac{\mathrm{e}^{-|\alpha|^2}}{2\pi\mathrm{i}} \oint_{|\zeta|=1} \mathrm{d}\zeta\; \frac{\mathrm{e}^{|\alpha|^2\zeta}}{\zeta^{\Delta+1}} \\
&=\; \frac{|\alpha|^\Delta \mathrm{e}^{-|\alpha|^2}}{\Delta\,!} = \mathrm{e}^{-k}\, k^k/k!\,.
\end{aligned} \tag{27}
$$

when we have used that the state is peaked on the constraint surface where $\Delta = |\alpha|^2 = k$.

The physical expectation value for the three Dirac observables is then,

$$
\langle \hat{L}_I\rangle_{\rm phy} := \frac{\langle\Psi_\alpha|\, \hat{L}_I|\Psi_\alpha\rangle_{\rm phy}}{\langle\Psi_\alpha|\Psi_\alpha\rangle_{\rm phy}} = L_I\big|_{\rm cl}\, \frac{k}{|\alpha_1|^2+|\alpha_2|^2}\;.
$$

Fluctuations.

For the fluctuations, on a physical coherent state we have,

$$\langle \hat{L}_I^2 \rangle_{\rm phy} = L_I^2|_{\rm cl} \, \frac{k(k-1)}{(|\alpha_1|^2 + |\alpha_2|^2)^2} + \frac{k}{4} \, . \qquad (28)$$

Therefore,

$$(\Delta \hat{L}_I)^2_{\rm phy} = L_I^2|_{\rm cl} \left(\frac{-k}{(|\alpha_1|^2 + |\alpha_2|^2)^2} \right) + \frac{k}{4} = -\frac{1}{k} \, L_I^2|_{\rm cl} + \frac{k}{4} \, . \qquad (29)$$

On the other hand, the fluctuations in the kinematical coherent states are given by

$$(\Delta \hat{L}_I)^2_{\rm kin} = \frac{1}{4} \, (|\alpha_1|^2 + |\alpha_2|^2) = \frac{\Delta}{4} \, . \qquad (30)$$

Thus, the difference between the fluctuations is given by

$$(\Delta \hat{L}_I)^2_{\rm kin} - (\Delta \hat{L}_I)^2_{\rm phy} = \frac{1}{k} \, L_I^2|_{\rm cl} < (\Delta \hat{C})^2_{\rm kin} \, . \qquad (31)$$

We conclude for this example:

i) The difference in the fluctuations is smaller than the fluctuation of the constraint operator on $\mathcal{H}_{\rm kin}$;

ii) Group averaging actually reduces the dispersions. If we begin with semi-classical kinematic states peaked at points on the constraint surface, physical states resulting from group averaging are *guaranteed* to be semi-classical. Furthermore, the kinematical calculation provides a good upper bound on the dispersion in the physical states.

Example 3: Constrained Energy Difference

$$\tilde{C} := \left(\frac{p_1^2}{2m} + kq_1^2\right) - \left(\frac{p_2^2}{2m} + kq_2^2\right) - \tilde{\Delta} = 0 \ . \quad (32)$$

We again have two harmonic oscillators with the same frequency, but now the constraint can be written as

$$C = |z_1|^2 - |z_2|^2 - \Delta = 0 \ , \quad (33)$$

The quantum constraint operator has the form

$$\hat{C} = (\hat{a}_1^\dagger \hat{a}_1 - \hat{a}_2^\dagger \hat{a}_2) - \tilde{\Delta} \ . \quad (34)$$

The physical states are then,

$$|\Psi_\alpha^{\mathrm{phy}}\rangle = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{d}\lambda \, \mathrm{e}^{\mathrm{i}\lambda\Delta} \, |(\alpha_1 \mathrm{e}^{-\mathrm{i}\lambda}), (\alpha_2 \mathrm{e}^{\mathrm{i}\lambda})\rangle \ . \quad (35)$$

which takes the form,

$$\|\Psi_\alpha^{\mathrm{phy}}\|^2 = \frac{\mathrm{e}^{-\Delta}}{2\pi\mathrm{i}} \oint_{|\zeta|=1} \frac{\mathrm{d}\zeta}{\zeta} \, \zeta^{-\Delta} \, \mathrm{e}^{(|\alpha_1|^2 \zeta + |\alpha_2|^2 \zeta^{-1})} . \quad (36)$$

The function to be integrated has a pole of infinite order at the origin, whence we can not compute the integral as easily in the previous example. But we can still express the result in terms of special functions.

$$\|\Psi^{\rm phy}_\alpha\|^2 = {\rm e}^{-(|\alpha_1|^2+|\alpha_2|^2)}|\alpha_1|^{2k} \sum_{m=0}^{\infty} \left[\frac{1}{k!\,(k+m)!}\right] |\alpha_1|^{2m}|\alpha_2|^{2m} \tag{37}$$

and use the identity

$$\sum_{n=0}^{\infty} \frac{(x/2)^{2n}}{n!\,(k+n)!} = \left(\frac{2}{x}\right)^k {\rm I}_k(x) \ . \tag{38}$$

where ${\rm I}_m$ is a modified Bessel function. Then, the norm can be expressed as:

$$\|\Psi^{\rm phy}_\alpha\|^2 = {\rm e}^{-(|\alpha_1|^2+|\alpha_2|^2)}(|\alpha_1|/|\alpha_2|)^k \ {\rm I}_k(2|\alpha_1||\alpha_2|) \ , \tag{39}$$

we again have three quadratic Dirac observables:

$$J_3 := \tfrac{1}{2}\,(z_1\bar{z}_1 + z_2\bar{z}_2) \ , \qquad J_+ := z_1\, z_2 \ , \qquad J_- := \bar{z}_1\, \bar{z}_2 \ ,$$

with their corresponding combinations, $J_1 = \frac{1}{2}(J_+ + J_-)$ and $J_2 = \frac{\rm i}{2}\,(J_+ - J_-)$. These observables provide a realization of the ${\rm sl}(2,R) = {\rm su}(1,1)$ Lie algebra. The expectation values of $\hat{J}_\pm$ —and therefore of $\hat{J}_{1,2}$— in the physical coherent states coincide with the classical values:

$$\langle \hat{J}_{1,2} \rangle_{\rm phy} = J_{1,2}|_{\rm cl} \ . \qquad (40)$$

For fluctuations we obtain

$$(\Delta \hat{J}_{1,2})^2_{\rm phy} = \frac{|\alpha_1||\alpha_2|}{4} \left[\frac{\mathrm{I}_{k-1}(2|\alpha_1||\alpha_2|) + \mathrm{I}_{k+1}(2|\alpha_1||\alpha_2|)}{\mathrm{I}_k(2|\alpha_1||\alpha_2|)} \right] + \frac{1}{2} \qquad (41)$$

and

$$(\Delta \hat{J}_{1,2})^2_{\rm kin} = \tfrac{1}{4} \left(|\alpha_1|^2 + |\alpha_2|^2 + 2 \right) . \qquad (42)$$

Therefore we need to compare them, In particular, the quotient $\frac{(\Delta J_{1,2})^2_{\rm phy} - (\Delta J_{1,2})^2_{\rm kin}}{(\Delta J_{1,2})^2_{\rm kin}}$
The fluctuations in both types of coherent states are of the same order. However, now the fluctuations are smaller in the kinematical coherent states than in the physical states.
But if the initial kinematical coherent states are chosen with tolerances $2\delta_i/3$, ϵ_i, the physical states will be semi-classical with desired tolerances δ_i and ϵ_i.

What to learn from this?

- For all the examples considered, by restricting the initial kinematical coherent states to have suitably small tolerances, the group averaged physical states can be guaranteed to be semi-classical for any specified choice of tolerances.
- The group averaging procedure offers a concrete and potentially powerful strategy to construct *physical* semi-classical states for a class of constrained systems.
- The purpose is to open a new avenue to investisgate wheter for more general systems, this property continues to hold.
- The strategy might work, or could serve as a guide for the search of the much desired states in semi-classical LQG.