"A full quantum theory and its symmetry reduction"

J. Engle
Institute for Gravitational Physics and Geometry, Penn State

Would like to thank A. Ashtekar for the suggestion which started this research and for many helpful discussions.

Outline.

- Introduction & motivation
- Brief review of quantization of free KG theory
- Look at different methods of imposing symmetry.
- Analysis of $\mathcal{H}_B$
- summarizing physical meaning of $\mathcal{H}_B$
- Application to LQG ....?

Introduction / Motivations

Purpose: To investigate appropriate notion of "symmetric sector" when relating a full theory to a corresponding symmetry reduced model.

A priori, there are two approaches to defining "symmetric sector":

(1.) The "symmetric sector" is the space of states fixed by the action of the symmetry group. (Impose symm. on <u>states</u>)

(2.) The "symmetric sector" is obtained by imposing symmetry constraints on the fields as a system of constraints à la Dirac (Impose symm. on <u>fields</u>)

We will argue (1.) is <u>not</u> the correct notion of "symmetric sector" when relating full theories to reduced models.

<u>Indeed</u>: in quantum gravity this <u>must</u> be the case as notion (1.) becomes trivial when diffeomorphism constraint is solved, whence it is a physically vacuous notion in q. grav.

In investigating these questions we will look at free KG field in Minkowski space with axisymmetry, & will look at 2 examples of notion (2.) above of symmetric sector.

Quantization of free KG theory: Review.

Classical: $\mathcal{C}$ = smooth fns $\varphi : \Sigma \to \mathbb{R}$ with appropriate fall-off.

$$\Gamma = T^*\mathcal{C} = \{(\varphi, \pi) \mid \varphi, \pi : \Sigma \to \mathbb{R}\}$$

$$\Rightarrow \quad \Omega([\varphi,\pi],[\varphi',\pi']) = \int_\Sigma (\pi\varphi' - \varphi\pi')\, d^3x$$

$$\mathbb{H} = \frac{1}{2}\int_\Sigma \left(\pi^2 + (\vec\nabla\varphi)^2 + m^2\varphi^2\right) d^3x$$

Prequantum structures:

$$J[\varphi,\pi] = [-\, H^{-\frac{1}{2}}\pi,\; H^{\frac{1}{2}}\varphi], \qquad H := -\Delta + m^2$$

complex structure from usual pos.-neg. freq. decomposition.

determines: $\langle \cdot, \cdot \rangle := \frac{1}{2}\Omega(J\cdot,\cdot) - \frac{i}{2}\Omega(\cdot,\cdot)$

"single-particle inner product."

Fock quantization

- $h :=$ completion of Γ w.r.t. $\langle \cdot , \cdot \rangle$
- $\mathcal{H} := \bigoplus_{n=0}^{\infty} \overset{n}{\otimes}_s h$
- For each $\xi = [\varphi, \pi] \in h$, we have the usual creation & annihilation operators $a^\dagger(\xi)$, $a(\xi)$.

- Representation of (smeared) field operators:

$$\hat{\varphi}[f] := i\{a([0,f]) - a^\dagger([0,f])\}$$

$$\hat{\pi}[g] := -i\{a([g,0]) - a^\dagger([g,0])\}$$

Imp. side note: these can be inverted:

$$a([f,g]) = \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g + if]$$

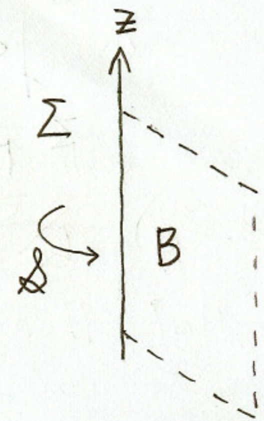
$$a^\dagger([f,g]) = \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f + ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g - if]$$

taking these expressions over to classical theory we obtain classical analogues of the creation & annihilation operators:

$$a([f,g]) = \tfrac{1}{2}\varphi[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\pi[\Theta^{-\frac{1}{2}}g + if] = \langle [f,g], [\varphi,\pi] \rangle$$

$$a^\dagger([f,g]) = \;---\; = \langle [\varphi,\pi], [f,g] \rangle \qquad \text{(functions on } \Gamma \ni (\varphi,\pi)\text{)}.$$

Reduced theory & its quantization.



- Γ_{inv} = axisymmetric subspace of Γ
- $\mathcal{C}_{inv}$ = axisymmetric subspace of $\mathcal{C}$
  $\cong$ smooth fns on $B := \Sigma/\mathcal{S}$ $(\cong \mathbb{R}^{+} \times \mathbb{R})$
  with sufficient fall off.

- $\Gamma_{inv} = T^{*}\mathcal{C}_{inv}$

since $\mathcal{C}_{inv}$ is a vector space & dynamics are linear, Fock quantization (& equiv. Schrödinger quantization) can again be done, with complex structure on Γ determined by the usual pos.-neg. freq.-decomp. rule.

Everything is done in the std. way, as if it were a theory just living on B.

[This is what we mean by the "reduced theory". Associated structures will be subscripted with "red".]

Different methods of imposing symmetry.

(1.) Invariance symmetry: $\hat{L}_z \Psi = 0$

Let space of sol'ns to this be denoted $\mathcal{H}_{inv}$

(2.) Imposing field-operator symmetries à la Dirac.

- Classical symmetry constraints: $\mathcal{L}_\phi \varphi(x) = 0$ and $\mathcal{L}_\phi \pi(x) = 0$.

  smeared: $\varphi[\mathcal{L}_\phi f] = 0$ and $\pi[\mathcal{L}_\phi g] = 0$.

- Problem: is 2^{nd} class system — cannot impose in quantum theory

  Reformulate as 1^{st} class system. Two natural ways:

  (A.) $\{\varphi[\mathcal{L}_\phi f] = 0\}_{f \in \mathcal{S}(\Sigma)}$

  (B.) $\{a[\mathcal{L}_\phi f, \mathcal{L}_\phi g] = 0\}_{f,g \in \mathcal{S}(\Sigma)}$

In the case $\Gamma = \{(q^i, p_i)\} = \mathbb{R}^{2n}$, $q^1 = 0$ & $p_1 = 0$ these reformulations are analogous to $\{q^1 = 0\}$ and $\{q^1 + i p_1 = 0\}$

Call these reformulations "A" and "B", and call the associated notions of symmetry "A-symmetry" and "B-symmetry."

Side note: Quantum mechanical implementation of A-symmetry leads to wave-fns w/ support only on symmetric configurations (when viewed from Schrödinger viewpoint).

Thus: A-symmetry is the analogue of Martin Bojowald's notion of symmetric states.

Analysis of B-symmetry.

Let $\mathcal{H}_B := \{\Psi \mid a([\mathcal{L}_\phi f, \mathcal{L}_\phi g])\Psi = 0 \quad \forall f, g \in \mathcal{S}(\Sigma)\}$

* 1.) $\mathcal{H}_B \overset{\text{nat.}}{\cong} \mathcal{H}_{red}$

2.) $\mathcal{H}_B \subsetneq \mathcal{H}_{inv}$

3.) $\mathcal{H}_B = \text{span}\{$coherent states associated with axisymmetric sector of classical theory$\}$

* 4.) $\mathcal{H}_B$ = space in which all "non-symmetric" modes are unexcited.

5.) Fluctuations from axisymmetry are balanced b/w configuration & momentum variables, & are minimized in a precise sense for states in $\mathcal{H}_B$
(For certain smearings f, $(\Delta_\Psi \hat{\varphi}[\mathcal{L}_\phi f])(\Delta_\Psi \hat{\pi}[\mathcal{L}_\phi f]) = \frac{1}{2}$, saturating Heisenburg bound).

* 6.) Quantum Hamiltonian $\hat{H}$ preserves $\mathcal{H}_B$

Note on A-symmetry : A-symmetry satisfies (1.) & (2.), but not (3.), (4.), (5.).
Not yet sure whether (6.) holds for A-symmetry, but I suspect not b/c classically the A-constraints do not weakly poisson-commute w/ the Hamiltonian.

---

*will explain these in more detail next.

$\mathcal{H}_B$ = space in which all "non-symmetric" modes are unexcited

More precisely

$$\mathcal{H}_B = \text{span}\,\{a^\dagger(\xi_1)\cdots a^\dagger(\xi_n)\,\Phi_0 \mid \xi_1, \ldots, \xi_n \in h_{inv}\}$$

or, alternatively use

→ $\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_\perp$

in terms of which

→ $\mathcal{H}_B = \{\psi \otimes 1 \mid \psi \in \mathcal{H}_{red}\}$.

One can show $\hat{H} = \hat{H}_{red} \otimes \mathbb{1} + \mathbb{1} \otimes \hat{H}_\perp$ for some $\hat{H}_\perp$ on $\mathcal{H}_\perp$.

1 is vacuum state of $\hat{H}_\perp$, whence $\mathcal{H}_B$ is space in which "all non-symmetric modes are unexcited."

can also immediately see:

- $\mathcal{H}_B \overset{\text{nat'l}}{\cong} \mathcal{H}_{red}$
- $\hat{H}$ preserves $\mathcal{H}_B$
  $\left(\hat{H}(\psi \otimes 1) = (\hat{H}_{red}\psi) \otimes 1 + \psi \otimes \overset{0}{\cancel{(\hat{H}_\perp 1)}} = (\hat{H}_{red}\psi) \otimes 1 \in \mathcal{H}_s\right)$

(as claimed).

Summary

$\mathcal{H}_B$ can be completely characterized in any of 3 diff. ways:

1.) $\mathcal{H}_B$ = solution space to a set of constraints whose class. analogues uniquely isolate the symmetric sector of the classical theory

2.) $\mathcal{H}_B$ = span of coherent states assoc. with symm. sector of classical theory

3.) $\mathcal{H}_B$ = space of states in which all non-symm. modes are unexcited.

In addition,
- Fluctuations from axisymmetry are under complete control in $\mathcal{H}_B$ & are in a certain sense minimized
- $\hat{H}$ preserves $\mathcal{H}_B$.

→ A-symmetry only shares 1 of the above 5 desireable properties with B-symmetry.

→ Invariance symmetry ($\mathcal{H}_{inv}$) doesn't achieve commutation of reduction & quantization even kinematically, has uncontrollable fluctuations from axisymmetry, & becomes physically vacuous in the case of quantum gravity any way.

conclusion: at least in KG case, B-symmetry is the best notion of symmetry when relating the full theory to the reduced theory. A-symmetry is not ideal but also works.

Quantum Gravity:

Bojowald embeds LQC into LQG via symmetry notion A.

Q: Is there an analogue of B-symmetry in LQG?

Suggestion:

→ Define B-symmetric sector to be the span of coherent states assoc. with symmetric sector of classical theory.

→ If you use complexifier coherent states, you will be reproducing characterizations (1.) & (2.) of last slide.

→ The free input of a complexifier might be used to adapt notion of symmetry to the Hamiltonian constraint s.t. Hamiltonian constraint preserves symmetric sector ....?

Just ideas. Could say more, but this is enough.

Misconception to be cleared up:

Sometimes people have a false intuition that there is no difference b/w the two notions of symmetry.

For example: consider quantum theory of point particle, $\mathcal{H} = L^2(\mathbb{R}^3)$, with symm. group: rotations about z-axis.

The two notions of symmetry are then:

(1.) Ψ is invariant under z-rotations

(2.) $\hat{L}_z \Psi = 0$.

But these are the same!

Answer: $L_z = 0$ is not a classical symmetry constraint. The classical theory in this case is that of a point particle. One doesn't talk about spatial symmetries of a point particle.

The 2nd notion of symmetry only applies in the field theory case.

Definition of $a^\dagger(\xi)$ and $a(\xi)$:

E.2

For $\Psi = (\psi, \psi^A, \psi^{A_1A_2}, \psi^{A_1A_2A_3}, \dots) \in \mathcal{H} \left(= \bigoplus_{n=0}^{\infty} \bigotimes_s^n h \right)$,

$$a^\dagger(\xi)\Psi := (0, \psi\xi^A, \sqrt{2}\,\xi^{(A_1}\psi^{A_2)}, \sqrt{3}\,\xi^{(A_1}\psi^{A_2A_3)}, \dots)$$

$$a(\xi)\Psi := (\bar{\xi}_A\psi^A, \sqrt{2}\,\bar{\xi}_A\psi^{AA_1}, \sqrt{3}\,\bar{\xi}_A\psi^{AA_1A_2}, \dots)$$

Schrödinger Quantization (is unitarily equiv. to Fock)

- $\overline{\mathcal{E}} = \mathcal{S}'(\Sigma)$, tempered distributions
- "$d\mu = \mathrm{Exp}\{-\frac{1}{2}(\varphi, \Theta^{\frac{1}{2}}\varphi)\}\mathcal{D}\varphi$", $\mathcal{D}\varphi$ = "Lesbegue measure on $\mathcal{S}'(\Sigma)$"

  (can be made into a rigorous measure on $\mathcal{S}'(\Sigma)$.)
- $\mathcal{H} = L^2(\overline{\mathcal{E}}, d\mu)$
- A "cylindrical function" on $\mathcal{S}'(\Sigma)$ is a function Ψ s.t.

  $\Psi[\varphi] = F(\varphi(e_1), \ldots, \varphi(e_n))$ for some smooth $F: \mathbb{R}^n \to \mathbb{C}$ & some $e_1, \ldots, e_n \in \mathcal{S}(\Sigma)$.
- Representation of (smeared) field operators:

  $$(\hat{\varphi}[f]\Psi)[\varphi] := \varphi[f]\,\Psi[\varphi]$$

  $$(\hat{\pi}[g]\Psi)[\varphi] := \left\{\text{self-adjoint part of } -i\int_\Sigma d^3x\, g\frac{\delta}{\delta\varphi}\right\}\Psi[\varphi]$$

  $$= -i\int_\Sigma d^3x \left(g\frac{\delta}{\delta\varphi} - \varphi\,\Theta^{\frac{1}{2}}g\right)\Psi[\varphi]$$
- Motivated by inverted eq'ns on slide 4, we define

  $$a([f,g]) := \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f - ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g + if]$$

  $$a^\dagger([f,g]) := \tfrac{1}{2}\hat{\varphi}[\Theta^{\frac{1}{2}}f + ig] + \tfrac{1}{2}\hat{\pi}[\Theta^{-\frac{1}{2}}g - if]$$
- Unique, normalized vacuum annihilated by all $a(\xi)$, $\xi \in h$:

  $$\Psi_0[\varphi] \equiv 1 .$$

The separation $\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_\perp$ in terms of the Schrödinger picture.

$$\overline{\mathcal{C}} = \mathcal{S}'(\Sigma), \quad \overline{\mathcal{C}}_{red} = \mathcal{S}'(B) \overset{nat.}{\cong} [\mathcal{S}'(\Sigma)]_{inv}$$

Let $\overline{\mathcal{C}}_\perp :=$ kernel of group averaging map $\Pi : \mathcal{S}'(\Sigma) \to [\mathcal{S}'(\Sigma)]_{inv}$.

Then one can show

$$\overline{\mathcal{C}} = \overline{\mathcal{C}}_{red} \oplus \overline{\mathcal{C}}_\perp$$

topologically, $\overline{\mathcal{C}} = \overline{\mathcal{C}}_{red} \times \overline{\mathcal{C}}_\perp$.

<u>Lemma</u>: $\exists$ measure $\mu_\perp$ on $\overline{\mathcal{C}}_\perp$ s.t.

$$\mu = \mu_{red} \otimes \mu_\perp$$

where μ is quantum measure in full theory & μ_{red} is quantum measure in reduced theory.

$\Rightarrow L^2(\overline{\mathcal{C}}, d\mu) = L^2(\overline{\mathcal{C}}_{red}, d\mu_{red}) \otimes L^2(\overline{\mathcal{C}}_\perp, d\mu_\perp)$

Let $\mathcal{H}_\perp := L^2(\overline{\mathcal{C}}_\perp, d\mu_\perp)$.

$\Rightarrow$ $\underline{\mathcal{H} = \mathcal{H}_{red} \otimes \mathcal{H}_\perp}$