On timelike faces
in
q-Lorentzian spinfoam models

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$\boxed{\text{Outline}}$

- Spinfoam models with $\Lambda = 0$
- Λ and quantum groups
- Towards a $\mathfrak{U}_q(su(1,1))$ model ?
- Conclusion

• GR $\equiv$ constrained BF theory

Idea : BF theory is topological $\Rightarrow$ direct quantization

$\longrightarrow$ Use this to quantize GR

• BF theory: $S_{BF} = \int_M Tr(B \wedge F[A])$, favorite group G, $\dim(M) = 4$

-Partition function: $Z_{BF}(M) = \int \mathcal{D}A \ \ \delta(F[A])$

-Discretization: triangulation $\mathcal{T}$ of M $\rightarrow$ dual 2-skeleton $=$ $(v, e, f) \subset \mathcal{T}^*$

$$\left.\begin{array}{l} A \rightarrow g_e \in G \\ F \rightarrow g_f = \prod_{e \in \partial f} g_e \end{array}\right\} \rightarrow Z_{BF}(\mathcal{T}) = \int \prod_e dg_e \prod_f \delta(g_f)$$

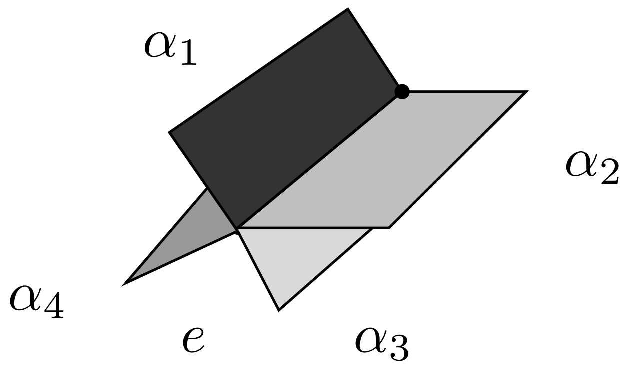
- Expansion of $\delta(g)$ in characters :

$$\rightarrow \mathcal{Z}_{BF}(\mathcal{T}) = \int \prod_e dg_e \prod_f \sum_{\{\alpha_f\}} \dim(\alpha_f) tr(\overset{\alpha_f}{\Pi}(g_f))$$

$\Rightarrow$ face $f \leftrightarrow$ representation α_f

-Calculation of the group integrals: each edge $e \leftrightarrow$ 4 faces $f \equiv$ 4 representations α_f

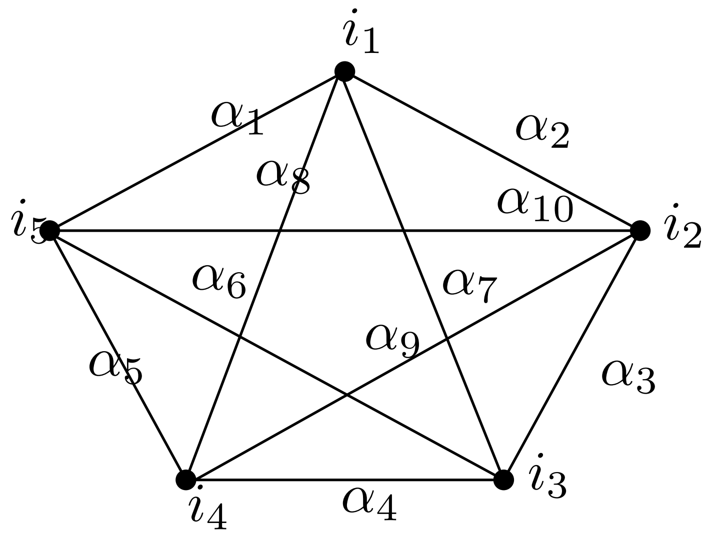
$\Rightarrow$ edge $e \leftrightarrow$ intertwiner $i_e \in Hom_G(\bigotimes_{f \in e} \overset{\alpha_f}{\mathbb{V}}, \mathbb{C})$



-Result: each vertex $v \equiv$ 10 faces & 5 edges

$\Rightarrow$ vertex $v \leftrightarrow$ 10 representations α_f contracted with 5 intertwiners $i_e \equiv$ $(10j)$ symbol $\equiv A(v)$

$\mathcal{Z}_{BF}(\mathcal{T}) = \sum_{\{\alpha_f\}} \sum_{\{i_e\}} \prod_f A(f) \prod_e A(e) \prod_v A(v)$



-Problem: Divergencies for large representations

• Gravity: $S = \int_M B^{IJ} \wedge F_{IJ}[A] - \frac{1}{2}\lambda_{IJKL} B^{IJ} \wedge B^{KL}$
$G = SL(2,\mathbb{C})_{\mathbb{R}}$

$\longrightarrow$ GR $\equiv$ BF + constraints on B

$\Rightarrow$ Idea: Impose the constraints at the simplicial level, i.e constrain B in the discretized BF path integral

-Sum over B configurations $\leftrightarrow$ sum over representations α's :

$$B_f = \int_f B \Leftrightarrow \alpha_f = (k_f, \rho_f) \in \mathbb{N}/2 \times \mathbb{R}$$

-BC Prescription to impose the constraints :

$\square$ faces f colored by simple representations ($C_2(k,\rho)$ =0) $\equiv$ $\alpha = (k,0)$ or $\alpha = (0,\rho)$ $\equiv$ area quantum number

$\rightarrow$ Face f colored by a representation $\alpha = (0,\rho)$ (resp $\alpha = (k,0)$) $\rightarrow$ spacelike (resp timelike)

☐ edges e labelled by BC intertwiners i_{BC} intertwining between simple representations

$$i_{BC} = \int_G dg(< \omega(\alpha_i) \mid \overset{\alpha_i}{\Pi}(g))^{\otimes_{i=1}^4}$$

- $\omega(\alpha)) \equiv su(2)$-invariant vector in $\overset{\alpha}{\mathbb{V}} \Leftrightarrow$ intertwiner between spacelike $\alpha = (0, \rho)$ faces

- $\omega(\alpha)) \equiv su(1,1)$-invariant vector in $\overset{\alpha}{\mathbb{V}} \Leftrightarrow$ intertwiner between spacelike $\alpha = (0, \rho)$ and timelike $\alpha = (k, 0)$ faces

☐ Result : Lorentzian BC model

$$\mathcal{Z}_{BC}(\mathcal{T}) = \sum_{\text{simple rep}} \prod_f A(f) \prod_e A(e) \prod_v A(v)$$

$\rightarrow A(v) \equiv (10j)(i_{BC})$

☐ Problem : $\sum_{\text{simple rep}} \equiv$ large area divergencies $\sim$ infrared

• Objective: Kill the infrared divergencies of the BC model

$\Rightarrow$ cut off on the representations

$\longrightarrow$ Physically: horizon $\Lambda > 0$

$\longrightarrow$ Mathematically: Switch group representations $\leftrightarrow$ quantum group representations

• How is Λ related to quantum groups ?

□ $Z_{BF,\lambda}(M) = \int \mathcal{D}A\mathcal{D}B \;\; e^{i\int_M Tr(B\wedge F) - \frac{\Lambda}{12} Tr(B\wedge B)}$

$\longrightarrow$ $Z_{BF,\Lambda}(M) \propto I_{CS}(\partial M)$

□ Crane-Yetter invariant $Z_{CY}(q,\mathcal{T}) \equiv 4d\ BF$ with $SU(2) \leftrightarrow \mathfrak{U}_q(su(2))$

$\longrightarrow$ $Z_{CY}(q,\mathcal{T}) = I_{CS}(\partial M)$

□ Result :

- $Z_{CY}(q, M) \propto Z_{BF,\Lambda}(M)$ with $q = \exp(il_p^2\Lambda)$
- q root of unity $\Rightarrow Z_{CY}(q, M)$ is finite

$\longrightarrow$ Well established relation $\Lambda \leftrightarrow$ quantum groups in the case of BF theory

$\Rightarrow \Lambda \leftrightarrow$ quantum groups $\Rightarrow$ finite models

• Idea: use the same methods for theories with local degrees of freedom $\Rightarrow$ cosmologically deform the BC model

☐ Noui-Roche model $\equiv$ 4d BC model *restricted to spacelike faces* with $SL(2,\mathbb{C})_{\mathbb{R}} \leftrightarrow \mathfrak{U}_q(sl(2,\mathbb{C}))_{\mathbb{R}}$

- $q = \exp(-l_p^2\Lambda) \Rightarrow$ bound on the area of a given face f:

$$l_p^2 \leq a(f) \leq 2\pi l_c^2$$

- $\mathcal{Z}_{NR}(q, \mathcal{T})$ is finite

☐ How to generalize such a model to include timelike faces ?

• What would be the basic building block of a q-BC model with spacelike and timelike components?

$\Rightarrow$ The BC intertwiner $\equiv$ a given representation module $\overset{\alpha}{\mathbb{V}}$ of the quantum Lorentz group and an invariant vector under $\mathfrak{U}_q(su(1,1))$

- First step: find an inclusion of $\mathfrak{U}_q(su(1,1))$ into the quantum Lorentz group

- Second step: represent the given $\mathfrak{U}_q(su(1,1))$ as to find the corresponding invariant vector ω

• Classically:

□ Lorentz group $\equiv$ classical double of $SU(2)$ (resp $SU(1,1)$)

- $sl(2,\mathbb{C})_{\mathbb{R}} = \mathcal{D}(su(2)) = su(2) \oplus su(2)^*$, with $SU(2)^* \simeq \mathcal{H}_3^+$

- $sl(2,\mathbb{C})_{\mathbb{R}} = \mathcal{D}(su(1,1)) = su(1,1) \oplus su(1,1)^*$

□ What can we use ?

$\Rightarrow$ A representation space $\overset{\alpha}{\mathbb{V}}$ such that

$$\mathcal{D}(su(2)) - \overset{\alpha}{\mathbb{V}} \equiv \text{finite, unitary action}$$

□ Idea: Construct $\gamma : \mathcal{D}(su(1,1)) \longrightarrow \mathcal{D}(su(2))$ such that $\gamma(\mathcal{D}(su(1,1)))$ can be represented unitarily

□ Result: γ gives us the classical inclusion of $su(1,1)$ inside the classical lorentz algebra as a star subalgebra

$\gamma(su(1,1)) - \overset{\alpha}{\mathbb{V}} \equiv$ finite action $\Rightarrow$ calculation of ω possible

- Quantum case:

□ Quantum Lorentz group $\equiv$ quantum double of $\mathfrak{U}_q(su(2))$ (resp $\mathfrak{U}_q(su(1,1))$)

- $\mathcal{D}\mathfrak{U}_q(su(2)) = \mathfrak{U}_q(su(2)) \otimes Pol(SU_q(2))^{op}$
- $\mathcal{D}\mathfrak{U}_q(su(1,1)) = \mathfrak{U}_q(su(1,1)) \otimes \mathfrak{U}_q(su(1,1))^*$

□ What have we got ?

$\Rightarrow$ A representation space $\overset{\alpha}{\mathbb{V}}$ such that

$$\mathcal{D}\mathfrak{U}_q(su(2)) - \overset{\alpha}{\mathbb{V}} \equiv \text{finite, unitary action}$$

□ We construct $\gamma : \mathcal{D}\mathfrak{U}_q(su(1,1)) \to \mathcal{D}\mathfrak{U}_q(su(2))$ such that γ gives us the quantum inclusion on $\mathfrak{U}_q(su(1,1))$ inside the quantum Lorentz group as a star subalgebra

$$\boxed{\gamma\ (\mathfrak{U}_q(su(1,1))) - \overset{\alpha}{\mathbb{V}} \equiv \ ?\ \Leftrightarrow \text{representation impossible}}$$

Conclusion

- First steps towards the definition of a q-Lorentzian model including spacelike and timelike faces
- Classical case $(\Lambda = 0)$ works
- Quantum case $(\Lambda > 0)$: technical obstructions
  - We have included $\mathfrak{U}_q(su(1,1))$ inside the quantum Lorentz group $\mathfrak{U}_q(sl(2,\mathbb{C})_{\mathbb{R}})$ as a star subalgebra
  - We can currently not represent the inclusion in any known representation module of the quantum Lorentz group
- Conclusions: ... we will continue working !