Stochastic Evolution of Graphs Using Local Moves

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LOOPS '05

Motivation
- Questions
- Definition of Local Moves

Outputs

Measurements

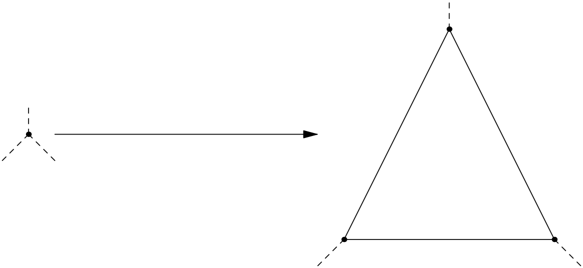
Conclusions

Questions

- Does a manifold-like structure result from the local application of Pachner moves?
- More generally, what are the properties of graphs generated by the local application of Pachner moves?

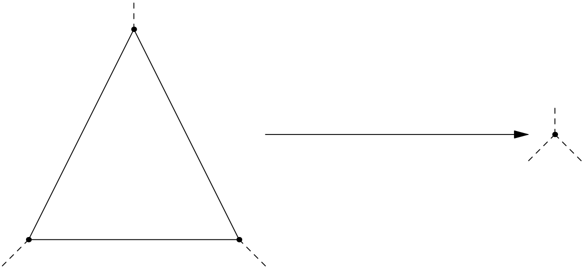
Expansion Move

An expansion move transforms a single vertex into a d-simplex (shown here for $d = 3$).



Contraction Move

A contraction move transforms a d-simplex into a single vertex (shown here for $d = 3$).



Exchange Move

An exchange move swaps the endpoints of two edges adjacent to some chosen edge.

Exchange Move and Embedding

Generalized exchange moves do not preserve planarity or any manifold-like structure.

Non-Local Edges

An edge is said to be “non-local” if it is not part of any short cycle (where short means the length of the cycle is $\leq r_l$). Intuitively, this should mean the edge connects locations in the graph which are “far away”.

Simulation Input Parameters

- d: The number of vertices in the simplex added during an expansion move.
- r_l: The length of the largest cycle to consider local.
- R: The ratio of exchange moves to expansion moves.
- R_c: The ratio of contraction moves to expansion moves.

Graph Pictures Key

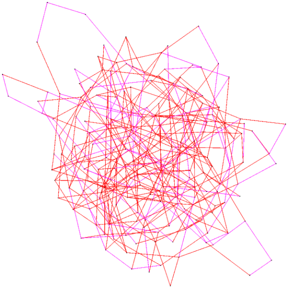
The following graph pictures were drawn using an algorithm which treats each edge as a spring with some spring constant and solves for the equilibrium configuration.

The edges are colored as follows (where $r_l = 5$):

- Red - Initial edges which are “local”
- Magenta - Initial edges which are “non-local”
- Black - Edges added by expansion moves which are “local”
- Green - Edges added by expansion moves which are “non-local”

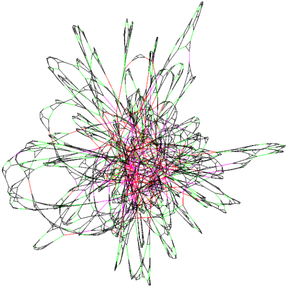
Initial Conditions

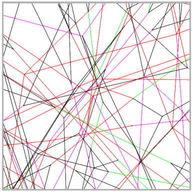
Initial conditions which were tried included a single triangle, regular lattices, almost complete graphs and random connected graphs (shown here with $n = 200$ and $m = 400$).

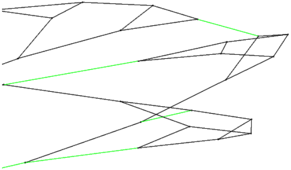


$R = 0$

With $R = 0$, the random graph has evolved to where $n = 2200$ and $m = 3400$. 14.7% of the edges are “non-local”.

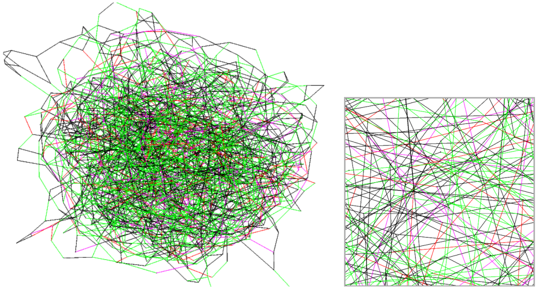






$R = 100$

With $R = 100$, the random graph has evolved to where $n = 1800$ and $m = 2800$. 42.5% of the edges are “non-local”.



Statistical Dimension

Let $V(r,v)$ be the number of vertices within a distance r of a vertex v. If $V(r,v) = c_v r^{d_v}$ where c_v and d_v are some constants then d_v is the statistical dimension with respect to the vertex v. The statistical dimension for the entire graph, d, is obtained by averaging d_v for all vertices.

Analytic Formula

Average vertex degree:

$$\bar{d} = \frac{2m}{n}$$

Number of vertices within a distance r ("volume"):

$$V(r) = V(r-1) + A(r)$$

Number of vertices not within a distance r:

$$R(r) = n - V(r-1)$$

Analytic Formula (cont.)

Number of “free” edges at a distance r (which connect vertices at a distance r to those at a distance $r+1$):

$$f(r) = (\bar{d} * A(r) - f(r-1))(1 - \frac{1}{2}(\frac{A(r-1)-1}{R(r)-1}))$$

Average number of “free” edges per vertex at a distance $r-1$:

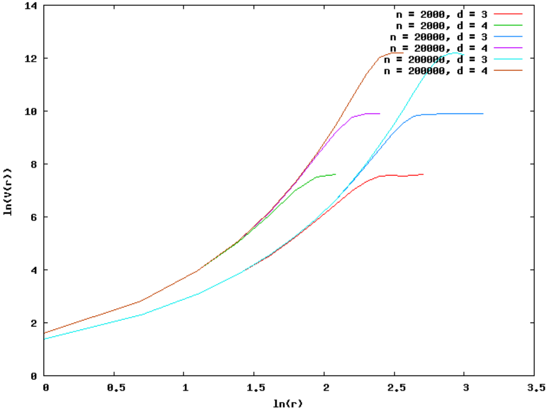
$$s(r) = f(r-1)/A(r-1)$$

The number of vertices at a distance r:

$$A(r) = R(r)(1 - (1 - \frac{s(r)}{R(r)})^{A(r-1)})$$

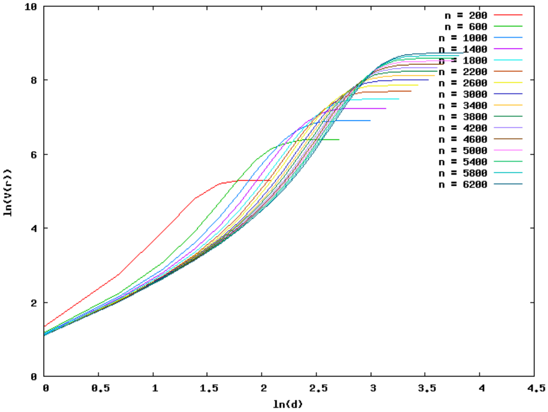
Analytic Formula Results

Plot of $\ln(V(r))$ vs. $\ln(r)$ for random connected graphs:



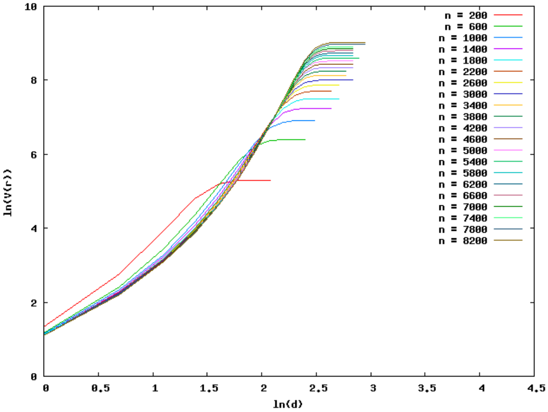
$R = 1$ Statistical Dimension Information

Plot of $\ln(V(r))$ vs. $\ln(r)$ for evolving graph with $R = 1$:



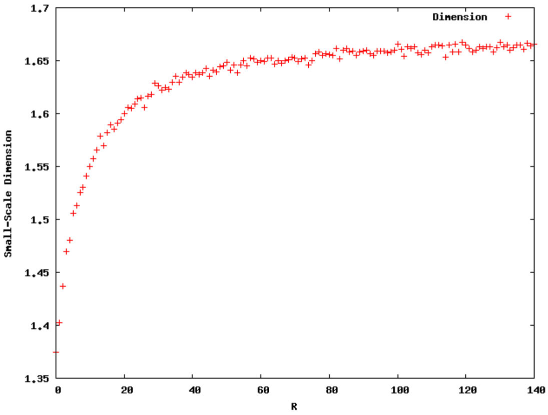
$R = 100$ Statistical Dimension Information

Plot of $\ln(V(r))$ vs. $\ln(r)$ for evolving graph with $R = 100$:



Small-Scale Dimension

Plot of small-scale dimension vs. R:

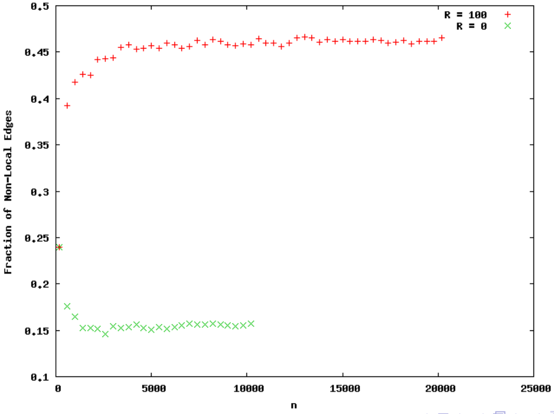


Small-Scale Dimension (cont.)

It seems that after some value of R the dynamics becomes "exchange dominated" whereby further increasing R does not significantly change the structure of the graph. This can also be seen using other measures (such as graph diameter).

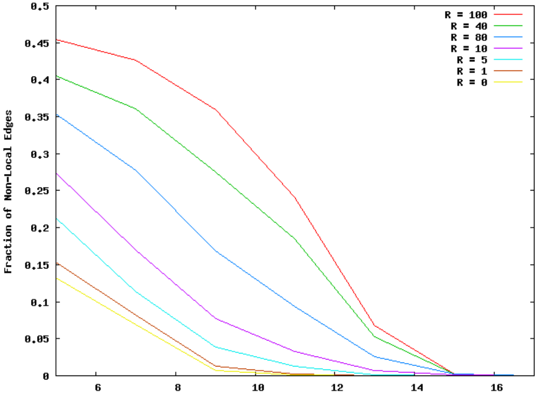
Asymptotic Fraction of Non-Local Edges

A plot of the fraction of non-local edges vs. *n*:



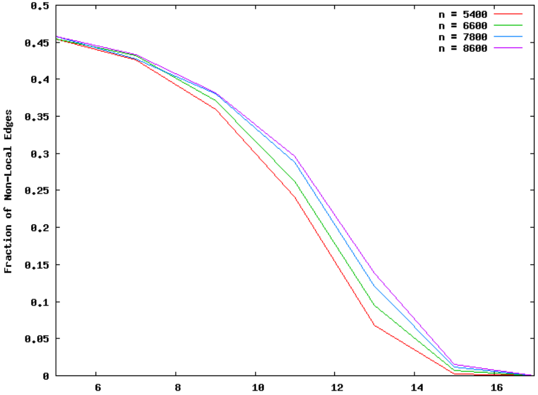
Effect of r_l

For a graph evolved to a fixed size $n = 5400$, a plot of the fraction of non-local edges vs. r_l:



Effect of r_l (cont.)

For a graph evolving at $R = 100$, a plot of the fraction of non-local edges vs. r_l:



Conclusions

- Stochastic application of local moves produces graphs which do not seem directly interpretable as approximating a smooth manifold. Thus, if graphs of similar structure represent space then it will be necessary to use some dynamic probe to characterize the approximated manifold.
- Unlabeled graphs will not suffice, additional degrees of freedom seem to be necessary in order to construct local moves which produce more manifold-like graphs.
- Macroscopic locality may be fundamentally non-local.

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