Stone-von Neumann Theorem in Quantum Geometry

Christian Fleischhack

Max-Planck-Institut für Mathematik in den Naturwissenschaften
Leipzig

Golm, October 2005

1 Contents

2 Motivation

- Quantization:
  representation of Poisson algebra by operators
  on kinematical Hilbert space

  Question: Free or unique choice?

- Standard Example: Quantum Mechanics

  Ingredients

configuration space $\mathcal{C}$	$\mathbb{R}$
measure on $\mathcal{C}$	$\mathrm{d}x$
kinematical Hilbert space	$L_2(\mathbb{R}, \mathrm{d}x)$

	position	momentum
selfadjoint	$\widehat{x} = x\,\cdot$	$\widehat{p} = -\mathrm{i}\partial_x$
exponentiated	$\mathrm{e}^{\mathrm{i}\sigma\widehat{x}} = \mathrm{e}^{\mathrm{i}\sigma x}\,\cdot$	$\mathrm{e}^{\mathrm{i}\lambda\widehat{p}} = L_\lambda^*$

Schrödinger representation

Why just these?

Weyl algebra $\mathfrak{A}$:

generated by 1-p groups $U(\sigma)$, $V(\lambda)$ with

$$U(\sigma)\,V(\lambda) \quad = \quad \mathrm{e}^{\mathrm{i}\sigma\lambda}\,V(\lambda)\,U(\sigma)$$

Stone-von Neumann Theorem 1930/1931

The Schrödinger representation is the only regular and irreducible representation of $\mathfrak{A}$.

3 Quantum Geometry: Basics

Gravity ... $SU(2)$ gauge field theory with constraints
(Gauß, Diffeo, Hamilton)

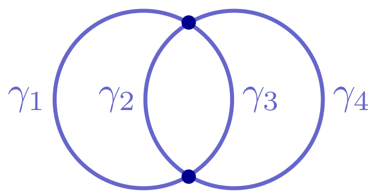
Quantum Geometry

- Generalized Connections

$$\overline{\mathcal{A}} \quad := \quad \mathrm{Hom}(\mathcal{P}, \mathbf{G}) \quad \ldots \quad \text{compact Hausdorff}$$

- Cylindrical Functions

$$\mathrm{Cyl} \quad := \quad \{f \in C(\overline{\mathcal{A}}) \mid f = f_{\boldsymbol{\gamma}} \circ \pi_{\boldsymbol{\gamma}}\}$$



with

$$\begin{array}{rcccl} \pi_{\boldsymbol{\gamma}} : & \overline{\mathcal{A}} & \longrightarrow & \overline{\mathcal{A}}_{\boldsymbol{\gamma}} & \equiv \mathbf{G}^n \\ & \overline{A} & \longmapsto & h_{\overline{A}}(\boldsymbol{\gamma}) & \equiv \big(h_{\overline{A}}(\gamma_1), \ldots, h_{\overline{A}}(\gamma_n)\big) \end{array}$$

- Ashtekar-Lewandowski Measure

μ_0 ... "pull-back" measure of Haar measures
i.e. $\pi_{\boldsymbol{\gamma}*}\mu_0 \quad = \quad \mu_{\text{Haar}}$

- Kinematical Hilbert Space

$$\mathcal{H} \quad := \quad L_2(\overline{\mathcal{A}}, \mu_0)$$

- Spin-Network Functions
  - · Spin network: graph $\boldsymbol{\gamma}$
    nontrivial irrep ϕ_k on each edge γ_k
  - · Spin-network function:

$$T_{\boldsymbol{\gamma},\boldsymbol{\phi}} \quad = \quad \bigotimes_k (\phi_k)^{i_k}_{j_k} \circ \pi_{\gamma_k} \quad : \quad \overline{\mathcal{A}} \longrightarrow \mathbb{C}$$

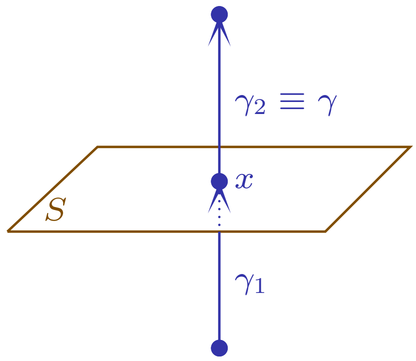
4 Weyl Algebra in Quantum Geometry

- Diffeomorphisms
  - Action: on graphs $\Longrightarrow$ on $\overline{\mathcal{A}}$ $\Longrightarrow$ on $C(\overline{\mathcal{A}})$
  - Example: $\alpha_\varphi(f \circ \pi_{\boldsymbol{\gamma}}) = f \circ \pi_{\varphi(\boldsymbol{\gamma})}$
  - μ_0 diffeoinvariant $\Longrightarrow$ diffeos act unitarily on $\mathcal{H}$
- Fluxes

$$h_{\Theta_d^S(\overline{A})}(\gamma) \;:=\; \underbrace{d}_{\text{function from } M \text{ to } \mathbf{G}}(\gamma(0))^{\kappa(\gamma,\underbrace{S}_{\text{hypersurface in } M})}\, h_{\overline{A}}(\gamma) \qquad (\text{for } \gamma(0) \in S)$$

with

$$\kappa(\gamma, S) = \begin{cases} +1 & (\gamma \text{ "above" } S) \\ 0 & (\text{otherwise}) \\ -1 & (\gamma \text{ "below" } S) \end{cases}$$



  - Example: parallel transports along $\gamma_1\gamma_2$

for $\overline{A}$	for $\Theta_d^S(\overline{A})$
$h_{\overline{A}}(\gamma_1)\, h_{\overline{A}}(\gamma_2)$	$h_{\overline{A}}(\gamma_1)\, d(x)^2\, h_{\overline{A}}(\gamma_2)$

  - Θ_d^S homeomorphism on $\overline{\mathcal{A}}$

- Weyl Operators

$$w_d^S := (\Theta_d^S)^* \;:\; C(\overline{\mathcal{A}}) \longrightarrow C(\overline{\mathcal{A}})$$

- Weyl Algebra for Quantum Geometry

$\mathfrak{A}$ ... C^*-subalgebra of $\mathcal{B}(L_2(\overline{\mathcal{A}}, \mu_0))$
generated by $C(\overline{\mathcal{A}})$ and $\mathcal{W} = \{w_d^S\}$

Goal: Representation Theory for $\mathfrak{A}$

5 Natural Representation π_0 of $\mathfrak{A}$ on $L_2(\overline{\mathcal{A}}, \mu_0)$

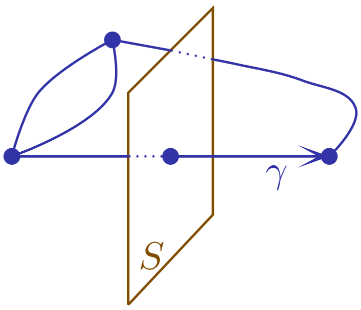
- Θ-invariance of $\mu_0 \quad \Longrightarrow \quad$ unitarity of $w = \Theta^*$
- $\pi_0 \quad \ldots \quad$ irreducible, regular, diffeo-invariant
- Irreducibility of π_0 CF: math-ph/0407006
  - $\mathfrak{A}' \;\subseteq\; C(\overline{\mathcal{A}})' \;=\; L_\infty(\overline{\mathcal{A}}, \mu_0)$
  - $w(f) = f \quad$ for all $w \in \mathcal{W}$ and $f \in \mathfrak{A}' \subseteq L_\infty$

    (by $w(f)w(\psi) = w(f\psi)$

    i.e. $w(f) \circ w = w \circ f$ in $\mathcal{B}(L_2)$)

    $$\Longrightarrow \quad \langle w(T), f\rangle = \langle T, f\rangle = \langle w'(T), f\rangle$$
  - γ labelled with irrep ϕ

    S labelled with constant $g \in \mathbf{G}$

    a) ϕ abelian:

    $$\langle T, f\rangle \;=\; \langle w(T), f\rangle \;=\; \overline{\phi(g^2)}\,\langle T, f\rangle$$

    b) ϕ nonabelian:

    $$\langle w_i(T), w_j(T)\rangle \;=\; \left|\frac{\chi_\phi(g^2)}{\dim \phi}\right|^2$$

    $$\Longrightarrow \quad \langle T, f\rangle = 0 \quad \text{for all } T \neq \mathbf{1} \text{ and } f \in \mathfrak{A}'.$$
  - Result: $\mathfrak{A}' = \mathbb{C}\,\mathbf{1}$

Expectation: Stone-von Neumann Theorem

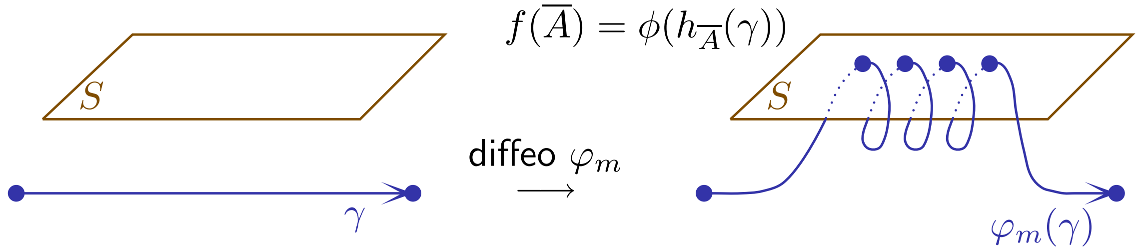
$$\pi \text{ irreducible, regular, diffeo-invariant} \quad \Longrightarrow \quad \pi \cong \pi_0$$

6 Stone-von Neumann Theorem

1. $\pi|_{C(\overline{\mathcal{A}})} = \bigoplus_\nu \pi_\nu$ canonical repr. of $C(\overline{\mathcal{A}})$ on $L_2(\overline{\mathcal{A}}, \mu_\nu)$

2. $\pi_\nu = \pi_0|_{C(\overline{\mathcal{A}})}$ for some ν

   Idea:



$$w^S_{\mathrm{e}^{t\mathfrak{d}}}(\alpha_{\varphi_m}(f)) \;=\; \phi(\mathrm{e}^{2mt\mathfrak{d}})\,\alpha_{\varphi_m}(f)$$

($\mathbf{G}$ abelian)

3. $\pi = \pi_0$
   - invert orientation of S by diffeo

     $\Longrightarrow$ w and w^* conjugate
   - assume: diffeo act naturally

     π_{ν_1} diffeo-invariant $\overset{\mu_1 = \mu_2}{\Longrightarrow}$ π_{ν_2} diffeo-invariant

Theorem: CF: math-ph/0407006

Assume
- $\dim M \geq 3$
- hypersurfaces – finitely triangulizable
- diffeos – stratified analytic
- diffeos act naturally
- d constant

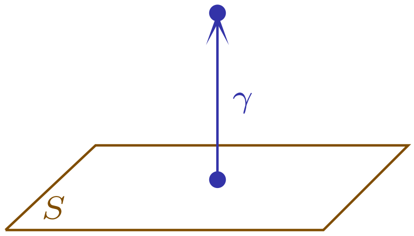
Then π_0 is the only regular representation of $\mathfrak{A}$ having a cyclic und diffeomorphism invariant vector.

7 Holonomy-Flux Algebra

- Smearing $\quad d = \mathrm{e}^{tf}$

$$h_{\theta^S_{tf}(\overline{A})}(\gamma) \quad := \quad \mathrm{e}^{\kappa(\gamma,S)\,t\,f(\gamma(0))}\; h_{\overline{A}}(\gamma) \qquad (\text{for } \gamma(0) \in S)$$

function from M to $\mathfrak{g}$



- Flux Vector Fields $\quad X^S_f : \mathrm{Cyl} \longrightarrow \mathrm{Cyl}$

$$X^S_f\,\psi \quad := \quad \frac{\mathrm{d}}{\mathrm{d}t}\Big|_{t=0}\, \psi \circ \theta^S_{tf}$$

  · Space of Generalized Vector Fields on $\overline{\mathcal{A}}$

$$\Gamma(T\overline{\mathcal{A}}) \quad := \quad \mathrm{Cyl} \cdot \langle X^S_f \rangle_{\text{Lie bracket}}$$

- ACZ Holonomy-Flux Algebra

$$\mathfrak{A}_{\mathsf{ACZ}} \quad := \quad \mathrm{Cyl} \times \Gamma(T\overline{\mathcal{A}})^{\mathbb{C}}$$

  with

$$\big\{(\psi_1, Y_1), (\psi_2, Y_2)\big\} \;=\; -\big(Y_1\psi_2 - Y_2\psi_1, [Y_1, Y_2]\big)$$

- Quantum Holonomy-Flux $*$-Algebra

$$\mathfrak{A}_{\mathsf{LOST}} \quad := \quad \mathrm{Free}_{\mathrm{lin}}(\mathfrak{A}_{\mathsf{ACZ}})/\text{relations}$$

  Relations:

$$\begin{aligned} (a,b) - (b,a) &\;=\; \mathrm{i}\,\{a,b\} && \text{(CCR)} \\ (\psi, c) + (c, \psi) &\;=\; 2\psi \cdot c && (\mathrm{Cyl}\ \text{module}) \end{aligned}$$

- Symmetries: $\quad$ analogous

Goal: $\quad$ States for $\mathfrak{A}$

8 States on $\mathfrak{A}_{\mathsf{LOST}}$

- Invariance w.r.t. bundle automorphisms $\varphi : P \longrightarrow P$

$$\omega \quad = \quad \omega \circ \alpha_\varphi$$

- Standard invariant state ω_0

$$\begin{array}{rcll} \omega_0(a \cdot \widehat{Y}) & = & 0 & (a \in \mathfrak{A}_{\mathsf{LOST}},\ Y \in \Gamma(T\overline{\mathcal{A}})^{\mathbb{C}}) \\ \omega_0(\widehat{\psi}) & = & \int_{\overline{\mathcal{A}}} \psi \, \mathrm{d}\mu_0 & (\psi \in \mathrm{Cyl}) \end{array}$$

Theorem: LOST: gr-qc/0504147

Assume
- $\dim M \geq 2$
- hypersurfaces – semianalytic
- diffeos – semianalytic
- smearings with compact support

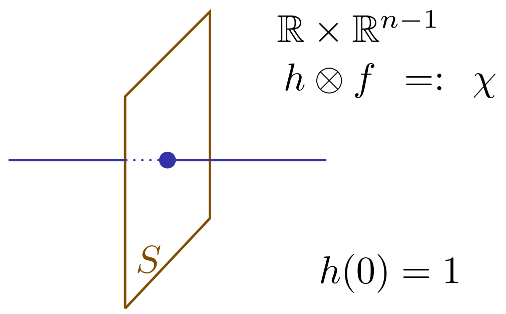
Then ω_0 is the only state on $\mathfrak{A}_{\mathsf{LOST}}$ that is invariant w.r.t. bundle automorphisms.

9 Proof of LOST Uniqueness Theorem

1. $[\widehat{X}_f^S] = 0$ locally

Idea:

$$\begin{aligned} (f_1, f_2) &:= \left\langle [\widehat{X}^S_{f_1 R}], [\widehat{X}^S_{f_2 R}] \right\rangle && (R \in \mathfrak{g}) \\ \varphi_\lambda &:= \mathrm{id} + \lambda \chi \vec{e} && \text{(diffeo for small } \lambda\text{)} \\ F(\vec{x}) &:= \vec{e} \cdot \vec{x} && \text{(on supp } \chi\text{)} \end{aligned}$$



$$\Longrightarrow \qquad \varphi_\lambda^* F = F + \lambda \chi$$

$$\begin{aligned} \Longrightarrow \qquad (F, F) &= (\varphi_\lambda^* F, \varphi_\lambda^* F) \\ &= (F, F) + 2\lambda \,\mathrm{Re}(F, f) + \lambda^2 \,(f, f) \end{aligned}$$

$$\Longrightarrow \qquad (f, f) = 0$$

2. $\mathcal{H}_\omega = L_2(\overline{\mathcal{A}}, \mu)$

- $\pi_\omega(\widehat{\psi}\, \widehat{Y}_1 \cdots \widehat{Y}_n)[\widehat{Y}_{n+1}]$ and $[\widehat{\psi}]$ generate $\mathcal{H}_\omega$

3. $\mu = \mu_0$

	LOST	Fleischhack
theory	gauge field theory	gauge field theory
geometric ingredients	principal fibre bundle P · structure group G · base manifold M	principal fibre bundle P · structure group G · base manifold M
smoothness	stratified analytic · $C^{\boldsymbol{k}}$ · semianalytic	stratified analytic · $C^{\boldsymbol{0}}$ · semi- **or subanalytic**
basic assumptions	· G compact connected Lie · M stratified analytic · $\dim M \geq \mathbf{2}$	· G compact connected Lie · M analytic · $\dim M \geq \mathbf{3}$
diffeomorphisms	stratified analytic	stratified analytic
positions · exponentiated · smeared along	connections · yes · paths	connections · yes · paths
paths	stratified analytic	stratified analytic
momenta · exponentiated · smeared along	fluxes · **no** · surfaces	fluxes · **yes** · surfaces
surfaces	stratified analytic · open · codimension 1 · —	stratified analytic · open · codimension 1+ · **widely triangulizable**
smearing functions	stratified analytic **compactly supported**	stratified analytic **constant on strata**
algebra	holonomy-flux algebra	Weyl algebra
type	$*$-algebra	$\boldsymbol{C^*}$-algebra
generators	positions · **cylindrical** functions on $\overline{\mathcal{A}}$	positions · **continuous** functions on $\overline{\mathcal{A}}$
	momenta · **weak derivatives of** pull-backs of left/right translations on $\overline{\mathcal{A}}$	momenta **(unitary)** · pull-backs of left/right translations on $\overline{\mathcal{A}}$
uniqueness	**state**	**representation**
assumed cyclicity	cyclic invariant vector	cyclic invariant vector
domain assumptions	**common dense domain: cylindrical functions**	—
regularity assumptions	—	**regularity w.r.t. smearing**
add'l assumptions	—	**natural diffeo action**
required invariance	**all** bundle automorphisms · diffeomorphisms · **gauge transformations**	some bundle automorphisms · some diffeomorphisms · —

Table 1: Comparison between LOST and Fleischhack