Consistency Check on Volume and Triad Operator Quantisation in Loop Quantum Gravity

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KG, T. Thiemann:	gr-qc/0507036 (main ideas and results)
	gr-qc/0507037 (technical details and proofs)
J. Brunnemann,T. Thiemann:	gr-qc/0405060 (matrix elements of volume operator)

Outline:
An Alternative Quantisation of the Flux Operator in LQG

1. Motivation of the concistency check
2. Short review of the usual flux operator in LQG
3. Idea and motivation of alternative quantisation
4. Construction of the alternative operator
5. Comparison between the action of the usual and the new flux operator
6. Conclusions

1. Motivation of the consistency check

- The dynamic of LQG is encoded in the Hamiltonian Constraint
- Classical:
$$C_{E}(N) = \int d^3x N C(x) =\propto \int N\,\mathrm{Tr}(F \wedge \mathbf{e})$$
- In order to define a corresponding expression in the quantum theory, one uses a Poisson bracket identity:
$$\mathbf{e}_{\mathbf{a}}^{\mathbf{k}} \propto \left\{A_a^k, \mathbf{V}(\mathbf{E})\right\}$$
- and promote operators via canonical quantisation
- This Poisson bracket identity has never been tested independently from the Hamiltonian constraint
- Exactly this can be done in defining an alternative flux operator!

2. Review of the Usual Flux Operator in LQG

Classical expression $\quad E_k(S) = \int\limits_S E^a_k n^S_a$

$$\left\{E_k(S), T_{\gamma,\vec{j},\vec{m}}(A)\right\} = \sum_{e\in E(\gamma)} \{E^a_k(S), (h_e)_{AB}\}\Big|_{h_{e'}=A(e')} \frac{\partial T_{\gamma,\vec{j},\vec{m}}}{\partial (h_e)_{AB}}$$

Smearing of S in order to compute the Poisson bracket $\quad \longrightarrow$ regularisation

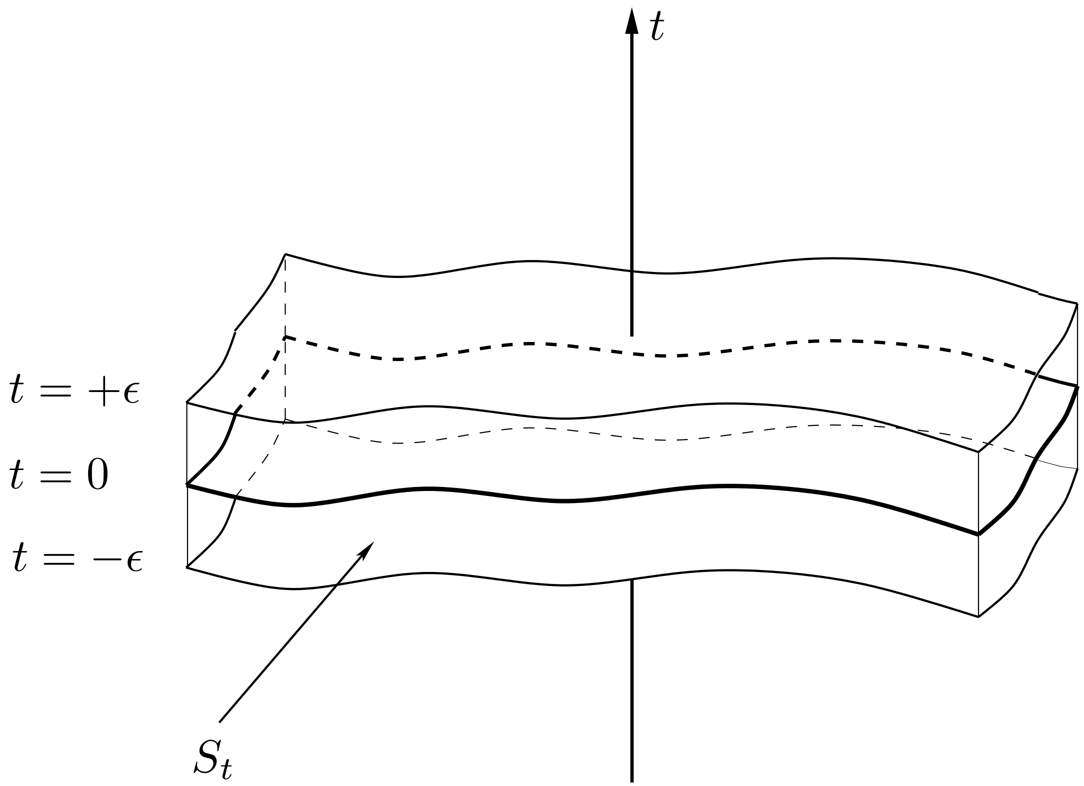
$$E^\epsilon_k(S) = \frac{1}{2\epsilon}\int\limits_{-\epsilon}^{\epsilon} dt\, E_k(S_t) \qquad \widehat{E}_k(S)\, T_{\gamma,\vec{j},\vec{m}} := \lim_{\epsilon\rightarrow 0} E^\epsilon_k(S)\, T_{\gamma,\vec{j},\vec{m}}$$

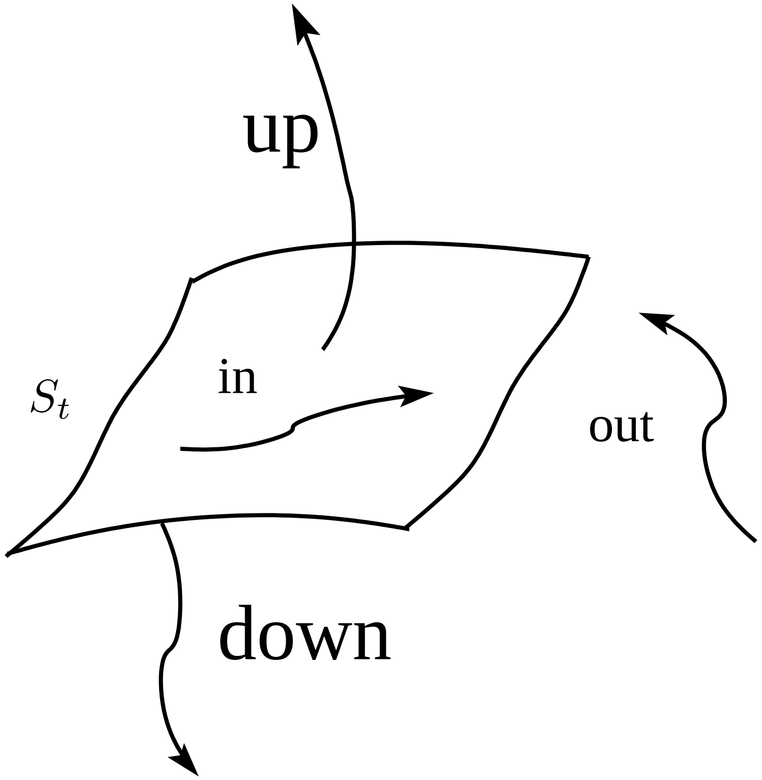
Action on an arbitrary SNF has to be investigated. Edges can be classified as (i)up (ii) down (iii) in and (iv) out wrt to S_t.

<u>Result</u>:

$$\widehat{E}_k(S)\, T_{\gamma,\vec{j},\vec{m}} = \frac{i}{2}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S)\left[\frac{\tau_k}{2} h_e\right]_{AB} \frac{\partial T_{\gamma,\vec{j},\vec{m}}(h_{e'}(A))_{e'\in E(\gamma)}}{\partial (h_{e'})_{AB}} = \frac{i}{4}\ell_p^2 \sum_{e\in E(\gamma)} \epsilon(e,S) X^e_k$$

$\epsilon(e,S) = \{-1, 0, +1\}$ depending on type of edge


t
$t = +\epsilon$
$t = 0$
$t = -\epsilon$
S_t


up
in
S_t
out
down

3. Alternative Flux Operator: Idea and Motivation

- Alternative regularized expression: $\widetilde{E}^{\epsilon}_k(S) = \frac{1}{2\epsilon}\int\limits_{-\epsilon}^{\epsilon} dt \boxed{\widetilde{E}_k(S_t)}$

- We take $\{A^j_a, E^b_k\} = \delta^3(x,y)\delta^a_b\delta^j_k$ as fundamental
- Two possible canonical transformations

$$
\begin{array}{llll}
I & A^j_a = \Gamma^j_a + \gamma\,\mathrm{sgn}(\det(e))K^j_a & , & E^a_j = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}e^s_b e^t_c \\
II & A^j_a = \Gamma^j_a + \gamma K^j_a & , & E^a_j = \frac{1}{2}\epsilon_{kst}\epsilon^{abc}e^s_b e^t_c\,\mathrm{sgn}(\det(e))
\end{array}
$$

- Two possibilities of choosing $\widetilde{E}_k(S_t) = \int\limits_{S_t} E^a_k n^{S_t}_a$, ($q_{ab} = e^i_a e^j_b \delta_{ij}$):

$$
E^a_k = \left\{ \begin{array}{lll} \det(e)e^a_k & = & \frac{1}{2}\epsilon_{kst}\epsilon^{abc}e^s_b e^t_c \\ \sqrt{\det(q)}e^a_k & = & \frac{1}{2}\epsilon_{kst}\epsilon^{abc}\underbrace{\mathrm{sgn}(\det(e))}_{=:\,\mathbf{S}}e^s_b e^t_c \end{array} \right\} =: \left\{ \begin{array}{c} \mathbf{E^{a,I}_k} \\ \mathbf{E^{a,II}_k} \end{array} \right\}
$$

- Instead of Quantising $E_k(S)$ as a differential operator, we express it classically in terms of triads e^a_k and use a different quantisation procedure for $\widetilde{E}_k(S)$

4. Construction of the Alternative Flux Operator

- Classical identity:

$$
\begin{aligned}
{}^{(\ell)}\widetilde{E}_k^{\mathbf{I},\mathbf{II}}(S_t) &= \lim_{\mathcal{P}_t \to S_t} \sum_{\square \in \mathcal{P}_t} \epsilon_{kst} \frac{4}{\kappa^2} \{A_3^s, V(R)\} \boxed{\mathbf{S}} \{A_4^t, V(R)\} \\
&= \lim_{\mathcal{P}_t \to S_t} \sum_{\square \in \mathcal{P}_t} \frac{16}{\kappa^2} \frac{1}{\frac{4}{3}\ell(\ell+1)(2\ell+1)} \\
&\quad \mathrm{Tr}\Big(\{\pi_\ell(h_{e_3}^{-1}), V(R)\} \, \pi_\ell(\tau_k) \boxed{\mathbf{S}} \pi_\ell(h_{e_4}) \, \{\pi_\ell(h_{e_4}^{-1}), V(R)\} \, \pi_\ell(h_{e_3}) \Big)
\end{aligned}
$$

consider arbitrary spin representation ℓ
usual canonical quantisation procedure $\longrightarrow$ Case I: $\widehat{V}^2$, Case II: $\widehat{V}\widehat{S}\widehat{V}$

- Now four possibilities since there exists $\widehat{V}_{AL}$ and $\widehat{V}_{RS}$:

$$
\begin{array}{llll}
{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{I}}(\mathbf{S_t}) & \longrightarrow & {}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{I,AL}}(\mathbf{S_t}) & {}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{I,RS}}(\mathbf{S_t}) \\
{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II}}(\mathbf{S_t}) & \longrightarrow & {}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,AL}}(\mathbf{S_t}) & {}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S_t})
\end{array}
$$

4. The two Volume Operators in LQG

- Definition of $\widehat{V}^{RS}$ (1994):

$$\widehat{V}^{RS}_{v,\gamma} = \sum_{I,J,K} \sqrt{\left| \frac{i}{8} C_{reg} \epsilon_{ijk} X^i_{e_I} X^j_{e_J} X^k_{e_K} \right|}$$

$\longrightarrow$ is NOT sensitive to the orientation of $\{e_I, e_J, e_K\}$

- Definition of $\widehat{V}^{AL}$ (1995):

$$\widehat{V}^{AL}_{v,\gamma} = \sqrt{\left| \frac{i}{8} C_{reg} \sum_{I,J,K} \epsilon(e_I, e_J, e_K) \underbrace{\epsilon_{ijk} X^i_{e_I} X^j_{e_J} X^k_{e_K}}_{\widehat{q}_{IJK}} \right|}$$

$\longrightarrow$ $\epsilon(e_I, e_J, e_K)$ sensitive to the orientation of $\{e_I, e_J, e_K\}$

- $\widehat{V}^{AL}$ sums INSIDE the square root over triples of edges
  $\widehat{V}^{RS}$ sums OUTSIDE the square root over triple of edges

- Factor ordering ambiguities: unique for AL; more than one for RS

4. What Do We Mean by Consistency:

- We have two flux operators: $\widehat{E}_k(S_t)$ and ${}^{(\ell)}\widehat{\widehat{E}}_k(S_t)$ in 4 different versions
- We want to apply both operators on the same SNF $|\beta^0\, {}_{m_{12}}>$
- The action on any other SNF can be derived from the action on $|\beta^0\, {}_{m_{12}}>$
- The difference between the action of these operators should only be a constant (at least semiclassically):

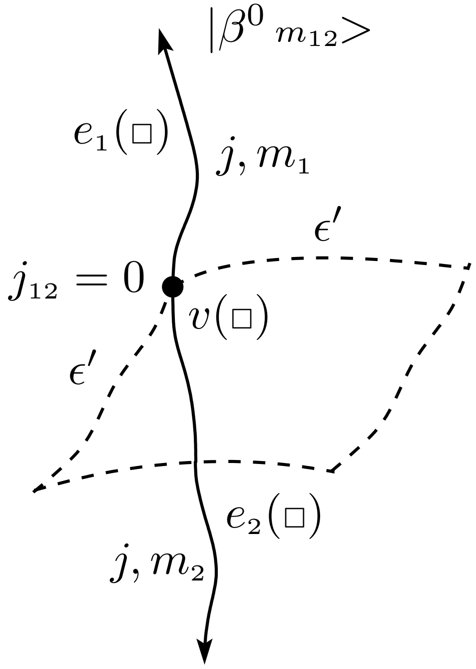
$$\sum_{\widetilde{m}_{12}} <\beta^1\, {}_{\widetilde{m}_{12}}|\, \widehat{E}_k(S_t)\, |\beta^0\, {}_{m_{12}}>\, |\beta^0\, {}_{\widetilde{m}_{12}}>=$$

$$C(j,\ell) \sum_{\widetilde{m}_{12}} <\beta^1\, {}_{\widetilde{m}_{12}}|\, {}^{(\ell)}\widehat{\widehat{E}}_{k,tot}(S_t)|\beta^0\, {}_{m_{12}}>\, |\beta^0\, {}_{\widetilde{m}_{12}}>$$

- This constant will fix the ambiguity C_{reg} of $\widehat{V}$ due to regularisation

4. Only the state $|\beta^0{}_{m_{12}}>$ is important

- Since for usual flux edges can be classified as edges of type up,down, in and out wrt to surface S, only the state $|\beta^0{}_{m_{12}}>$ is of interest!



4. Results in the Case ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{AL}}_{\mathbf{k}}(\mathbf{S_t})$ and ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{RS}}_{\mathbf{k}}(\mathbf{S_t})$

- Results for ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{AL}}_{\mathbf{k}}(\mathbf{S_t})$:
  Calculation can only be done for $\ell = 0.5, 1$ analytically.
  $< \beta^1 \, \widetilde{m}_{12} \, |{}^{(\ell)}\widehat{\widetilde{\mathbf{E}}}^{\mathbf{I},\mathbf{AL}}_{\mathbf{k},}(\mathbf{S_t})| \, \beta^0 \, m_{12} >= 0$
- Results for ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{RS}}_{\mathbf{k}}(\mathbf{S_t})$:
  Calculation can only be done for $\ell = 0.5, 1$ analytically.
  $< \beta^1 \, \widetilde{m}_{12} \, |{}^{(\ell)}\widehat{\widetilde{\mathbf{E}}}^{\mathbf{I},\mathbf{RS}}_{\mathbf{k}}(\mathbf{S_t})| \, \beta^0 \, m_{12} >= 0$
  (including the addional terms due to the commutator)
- Both versions of the volume operator lead to <u>non consistent</u> flux operators ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{AL}}_{\mathbf{k}}(\mathbf{S_t})$ and ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{I},\mathbf{RS}}_{\mathbf{k}}(\mathbf{S_t})$
- $\longrightarrow$ Case **I** where $\widetilde{\mathbf{E}}^{\mathbf{a},\mathbf{I}}_{\mathbf{k}} = \det(e) e^a_k$ is ruled out!

5. Case II: The Signum Operator $\widehat{\mathbf{S}}$

- Case II, classical: $\mathbf{S} := \mathrm{sgn}(\det(e)) = \mathrm{sgn}(\det(e)\underbrace{\det(q)}_{>0}) = \mathrm{sgn}(\det(E))$
- Connection the $\widehat{V}^{AL}$ operator:

$$\widehat{V}^{AL}_{v,\gamma} = \sqrt{\left|\frac{i}{8} C_{reg} \sum_{I,J,K} \epsilon(e_I, e_J, e_K)\epsilon_{ijk} X^i_{e_I} X^j_{e_J} X^k_{e_K}\right|} =: \sqrt{|\widehat{Q}|}$$

- One can show classically $\mathbf{S} = \mathrm{sgn}(Q)$, hence $\widehat{\mathbf{S}}^{\mathbf{AL}} = \mathrm{sgn}(\widehat{Q})$ and $\widehat{Q} = \widehat{V}^{AL}\widehat{\mathbf{S}}^{\mathbf{AL}}\widehat{V}^{AL}$
- NOT possible for $\widehat{V}^{RS}$

$$\widehat{V}^{RS}_{v,\gamma} = \sum_{I,J,K} \sqrt{\left|\frac{i}{8} C_{reg} \epsilon_{ijk} X^i_{e_I} X^j_{e_J} X^k_{e_K}\right|}$$

since NOT sensitive to orientation of edges and NO single square root, but sum of single square roots

5. Results in the Case ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{k}}(\mathbf{S_t})$

- The matrix element $<\beta^1\,|\,{}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{k}}(\mathbf{S_t})\,|\,\beta^0>$ is non zero!
- Comparison between the two operators ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{k}}(\mathbf{S_t})$ and $\widehat{E}_k(S)$:

$$
\begin{aligned}
{}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{1}}(\mathbf{S})|\,\beta^0\;{}_0> &= \boxed{48\,C_{reg}}\,\widehat{E}_1(S)|\,\beta^0\;{}_0> \\
{}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{2}}(\mathbf{S})|\,\beta^0\;{}_0> &= \boxed{48\,C_{reg}}\,\widehat{E}_2(S)|\,\beta^0\;{}_0> \\
{}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{3}}(\mathbf{S})|\,\beta^0\;{}_0> &= \boxed{48\,C_{reg}}\,\widehat{E}_3(S)|\,\beta^0\;{}_0>
\end{aligned}
$$

- Therefore if we choose $C_{reg}=\frac{1}{48}$ the results of the two operators agree (AL!)
- ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II,AL}}_{\mathbf{1}}(\mathbf{S}) \propto \widehat{V}^{AL}\widehat{\mathbf{S}}^{\mathbf{AL}}\widehat{V}^{AL}$ leads to a consistent result with the usual flux operator $\widehat{E}_k(S)$
- Exact agreement, not only semiclassically! Result independent of spin label ℓ!

5. Results in the Case ${}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S_t})$

- It is impossible to quantise $\widehat{\mathbf{S}}$ in the same way as $\widehat{V}^{RS}$ is quantised
  $\longrightarrow$ conclusion $\widehat{\mathbf{S}}^{\mathbf{AL}}$ does not exist$\longrightarrow$ ${}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S_t})$ is therefore inconsistent with the usual $\widehat{E}_k(S)$
- Use ${}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S_t}) \propto \widehat{V}^{RS}\widehat{\mathbf{S}}^{\mathbf{AL}}\widehat{V}^{RS}$ (somehow "artificial" construction)
- Result: The matrix element $<\beta^1\,|\,{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S_t})\,|\,\beta^0>$ is non zero
- BUT: Comparison between the two operators ${}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{k}}^{\mathbf{II,RS}}(\mathbf{S})$ and $\widehat{E}_k(S)$:

$$
\begin{aligned}
{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{1}}^{\mathbf{II,RS}}(\mathbf{S_t})|\,\beta^0\ 0> &= \boxed{C(j,\ell)\,C_{reg}}\widehat{E}_1(S)|\,\beta^0\ 0> \\
{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{1}}^{\mathbf{II,RS}}(\mathbf{S_t})|\,\beta^0\ 0> &= \boxed{C(j,\ell)\,C_{reg}}\widehat{E}_2(S)|\,\beta^0\ 0> \\
{}^{(\ell)}\widetilde{\mathbf{E}}_{\mathbf{1}}^{\mathbf{II,RS}}(\mathbf{S_t})|\,\beta^0\ 0> &= \boxed{C(j,\ell)\,C_{reg}}\widehat{E}_3(S)|\,\beta^0\ 0>
\end{aligned}
$$

- Semiclassically: $C(j,\ell) \rightarrow C(\ell)$
  ℓ-dependence of $C(j,\ell)$ is due to additional linearly dependent triples!
  $\longrightarrow$ modification by hand?
- Conclusion: The volume operator $\widehat{V}^{RS}$ is inconsistent with the usual $\widehat{E}_k(S)$

6. Conclusions

- We managed to implement an alternative flux operator in 4 different versions
- The result of ${}^{(\ell)}\widetilde{\mathbf{E}}^{\mathbf{II},\mathbf{AL}}_{\mathbf{k}}(\mathbf{S_t})$ does agree with the usual flux operator, if we choose $C_{reg} = \frac{1}{48}$
- $C_{reg} = \frac{1}{48}$ is exactly the regularisation constant found by Ashtekar and Lewandowski
- The Poisson bracket identity has been tested independently and is consistent with LQG (surprising discrimination between $\widehat{V}$ not anticipated)
- We could rule out $\widetilde{\mathbf{E}}^{\mathbf{a},\mathbf{I}}_{\mathbf{k}} = \det(e)e^a_k$, i.e. $\widetilde{\mathbf{E}}^{\mathbf{a},\mathbf{II}}_{\mathbf{k}}$ has to be a pseudo two form
- Restriction to $\det(E) \geq 0$ problem with momentum operator!
- technically non trivial check, result could not have been guessed from the outset