C. Meusburger, Perimeter Institute for Theoretical Physics

Grafting and Poisson structure in (2+1)-gravity with $\Lambda = 0$

Loops '05, Golm, October 10-14 2005

References:

1. C. Meusburger, **Grafting and Poisson structure in (2+1)-gravity with vanishing cosmological constant**, gr-qc/0508004

2. C. Meusburger, B. J. Schroers: **Mapping class group actions in Chern-Simons theory with gauge group $G \ltimes \mathfrak{g}^*$**, Nucl. Phys. B 706 (2005) 569-597, hep-th/0312049

Outline

Aim: understand relation between geometrical and gauge theoretical formulation of (2+1)-dimensional gravity with $\Lambda = 0$

absence of local gravitational degrees of freedom

geometrical formulation	Chern-Simons gauge theory
flat metric spacetimes $M = U/\pi_1(M)$, $U \subset \mathbb{M}^3$	flat gauge field
+ physical interpretation	+ description of phase space + complete set of observables
- description of phase space ?	- reconstruction of geometry ? - physical interpretation ?

Question: geometrical interpretation of observables and associated transformations on phase space?

Result: 2 basic observables for closed, simple curves in spacetime generate 2 basic geometry transformations: grafting and Dehn twists

Contents:

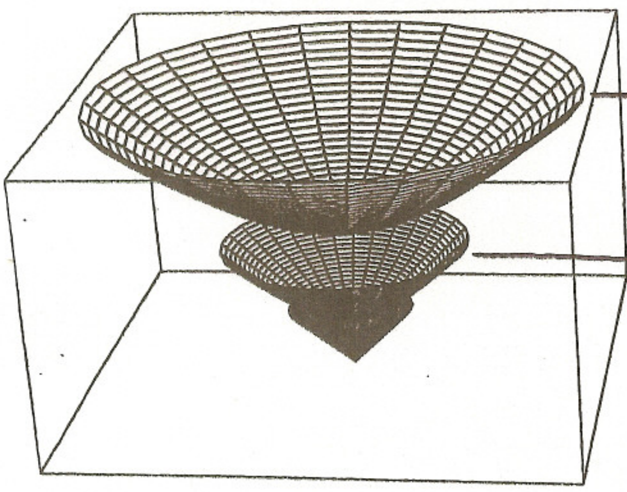
1 Construction of (2+1)-spacetimes via grafting

[Benedetti, Bonsante, Guadagnini]

spacetimes: $M \approx \mathbb{R} \times S_g,\ g \geq 2 \quad \Rightarrow \quad M = U/\pi_1(S_g),\ U \subset \mathbb{M}^3$

1.1 Static spacetimes

1. Foliate interior of lightcone U by hyperboloids H_T



hyperboloid

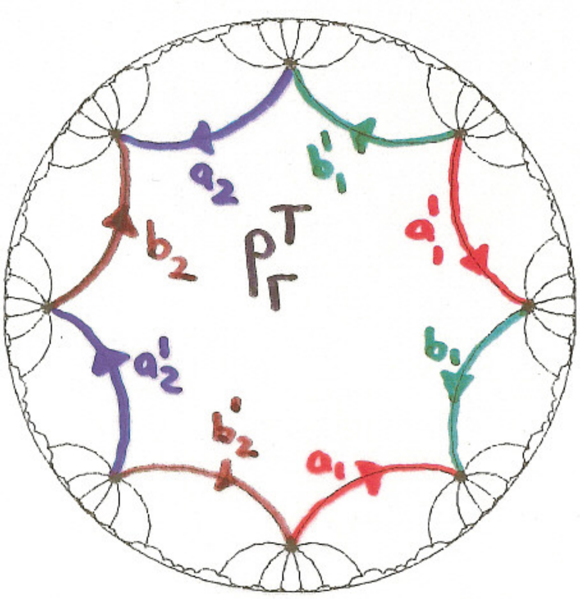
$H_T = \{x \in \mathbb{M}^3 \mid x_0^2 - x_1^2 - x_2^2 = T^2,\ x^0 > 0$

hyperbolic space H_1

2. Cocompact Fuchsian group Γ

$$SO(2,1) \supset \Gamma = \langle v_{A_1}, v_{B_1}, \ldots, v_{A_g}, v_{B_g}\,;\, [v_{B_g}, v_{A_g}^{-1}] \cdots [v_{B_1}, v_{A_1}^{-1}] = 1 \rangle \cong \pi_1(S_g)$$

$\Rightarrow$ tesselation of H_T by geodesic arc $4g$-gons



fundamental polygon $P_\Gamma^T \subset H_T$

generators of Γ: $v_{A_i} : a_i \mapsto a_i',\ v_{B_i} : b_i \mapsto b_i'$

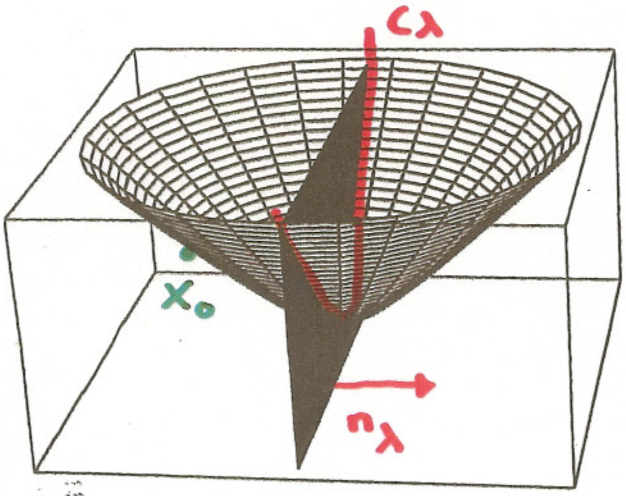
3. spacetime: $M = U/\Gamma$

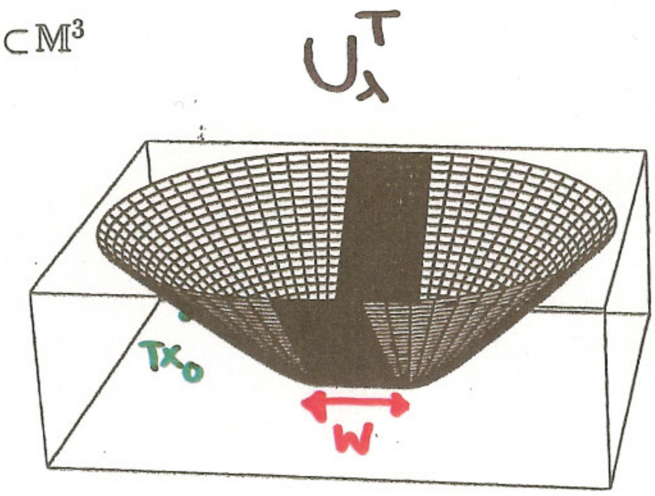
1.2 Grafting

ingredients: Γ, closed simple geodesic λ on $S_\Gamma = H_1/\Gamma$ with weight $w > 0$

1. λ lifts to Γ-invariant multicurve $M_\lambda = \{vc_\lambda | v \in \Gamma\} \subset H_1$

2. Grafting: $M_\lambda \Rightarrow$ regular domain $U_\lambda \subset \mathbb{M}^3$





   - choose basepoint outside geodesics
   - cut along planes associated to geodesics
   - translate away from basepoint along plane's normal vector, distance w
   - join pieces by straight lines
   - domain: $U_\lambda = \bigcup_{T \in \mathbb{R}_0^+} U_\lambda^T$, U_λ^T = surfaces of constant cosm. time T

3. Action of $\Gamma \cong \pi_1(S_g)$ on U_λ: U_λ^T invariant, free, properly discontinuous

4. Grafted spacetime: $M = U_\lambda/\Gamma$

2 Phase space and Poisson structure in the Chern-Simons formulation

gauge group: $ISO(2,1) = SO(2,1) \ltimes \mathbb{R}^3$:

- generators $J_a, P_a \in iso(2,1)$: $[J_a, J_b] = \epsilon_{abc} J^c$ $[J_a, P_b] = \epsilon_{abc} P^c$ $[P_a, P_b] = 0$

- parametrisation: $(u, \boldsymbol{a}) = (u, -u\boldsymbol{j})$, $u = e^{-p^a J_a} \in SO(2,1)$, $\boldsymbol{a}, \boldsymbol{j} \in \mathbb{R}^3$
  $(u_1, \boldsymbol{a}_1)(u_2, \boldsymbol{a}_2) = (u_1 u_2, \boldsymbol{a}_1 + u_1 \boldsymbol{a}_2)$

- formal parameter θ, $\theta^2 = 0 \Rightarrow$ representation $(P_a)_{bc} = \theta (J_a)_{bc} = -\theta \epsilon_{abc}$
  $(u, \boldsymbol{a}) \leftrightarrow (1 + \theta a^b J_b) u$

gauge field: $A = e^a P_a + \omega^a J_a = A_0 dx^0 + A_S$

equations of motion:

$$F_S = d_S A_S + A_S \wedge A_S = 0 \quad \partial_0 A_S = d_S A_0 + [A_S, A_0]$$

observables for $\lambda \in \pi_1(S_g)$:

conjugation invariant functions of holonomy $H_\lambda = (e^{-p_\lambda^a J_a}, -e^{-p_\lambda^a J_a} \boldsymbol{j}_\lambda)$

$$m_\lambda^2 = -\boldsymbol{p}_\lambda^2 \quad m_\lambda s_\lambda = \boldsymbol{p}_\lambda \boldsymbol{j}_\lambda$$

phase space:

parametrised by holonomies A_i, B_i of generators a_i, b_i of $\pi_1(S_g)$

$\mathcal{M}_g = \{(A_1, B_1, \ldots, A_g, B_g) \in ISO(2,1)^{2g} \mid [B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] = 1\} / ISO(2,1)$

Poisson structure: from symplectic potential on $ISO(2,1)^{2g}$

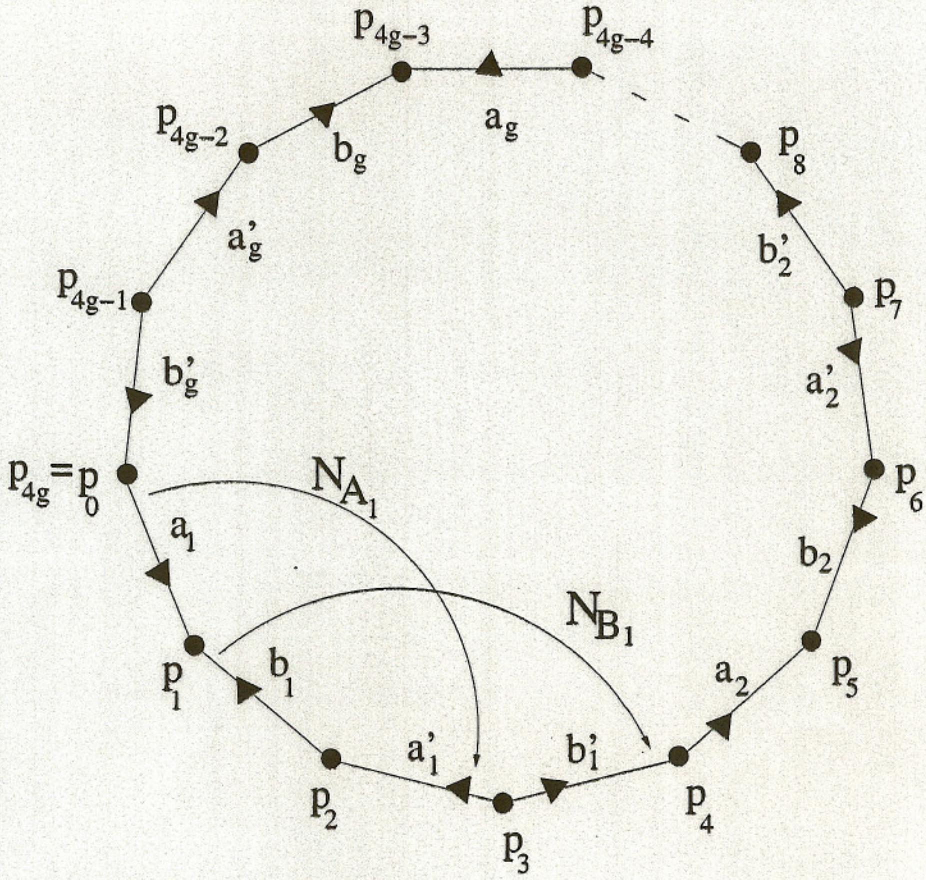
by imposing the constraint $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] = 1$ and dividing by the associated gauge transformations (simultaneous conjugation with $ISO(2,1)$

Trivialisation and embedding

trivialisation: on simply connected region $R \subset \mathbb{R} \times S_g$:

$$A_S = \gamma d_S \gamma^{-1}, \quad \gamma = (v, \boldsymbol{x}) : R \to ISO(2,1)$$
$$\boldsymbol{x} : R \to \mathbb{R}^3 = \text{embedding into } \mathbb{M}^3$$

maximal simply connected region by cutting S_g along the generators $a_i, b_i \in \pi_1(S_g) \Rightarrow 4g$-gon P_g



overlap condition:

$\gamma^{-1}|_{a_i'} = N_{A_i}\gamma^{-1}|_{a_i} \quad \gamma^{-1}|_{b_i'} = N_{B_i}\gamma^{-1}|_{b_i}$ with constants $N_{A_i}, N_{B_i} \in ISO(2,1)$
$\Rightarrow$ determined completely by embedding of sides a_i, a_i', b_i, b_i'

holonomies: $A_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-4}) \qquad B_i = \gamma(p_{4i-3})\gamma^{-1}(p_{4i-2})$

3 Grafting in the Chern-Simons formalism

idea: identify $x^0\!=\!T$

$\Rightarrow$ polygon P_g embedded on surfaces U_λ^T of constant cosmological time T

$\Rightarrow$ determine holonomies from embedding of sides of P_g

static case: polygon P_g embedded onto polygon P_Γ^T in tesselation of H_T

$$\boldsymbol{x}_{st}(T,\cdot) : P_g \mapsto P_\Gamma^T \subset H_T \Rightarrow \quad N_{A_i}^{st} = (v_{A_i}, 0), N_{B_i}^{st} = (v_{B_i}, 0)$$

grafted spacetime: $\boldsymbol{x}(T,\cdot) : P_g \to U_\lambda^T \quad \Rightarrow \quad N_{A_i}\!=\!(v_{A_i}, ?),\; N_{B_i}\!=\!(v_{B_i}, ?)$

$\Rightarrow$ translation of corners: $\quad \boldsymbol{x}(T, p_i) = \boldsymbol{x}_{st}(T, p_i) + \rho(p_i)$

transformation of holonomies under grafting along λ

$$Gr_{w\lambda} : {A_i}^{st} \mapsto {A_i}^{st}(1, \rho(p_{4i-4}) - \rho(p_{4i-3})) \quad {B_i}^{st} \mapsto {B_i}^{st}(1, \rho(p_{4i-2}) - \rho(p_{4i-3}))$$

4 Grafting and Poisson structure

Theorem The grafting transformation $Gr_{w\lambda} : ISO(2,1)^{2g} \to ISO(2,1)^{2g}$ is generated via the Poisson bracket by the mass m_λ

$$F \circ Gr_{w\lambda} = -\{wm_\lambda, F\} \qquad \forall F \in C^\infty(ISO(2,1)^{2g}).$$

$\Rightarrow$ **Properties of the grafting transformation $Gr_{w\lambda}$:**

1. Poisson isomorphism $\{F \circ Gr_{w\lambda}, G \circ Gr_{w\lambda}\} = \{F, G\} \circ Gr_{w\lambda}$
2. Leaves constraint $[B_g, A_g^{-1}] \cdots [B_1, A_1^{-1}] \approx 1$ invariant and commutes with the associated gauge transformations by simultaneous conjugation
3. Grafting transformations $Gr_{w\lambda}$ for different $\lambda \in \pi_1(S_g)$ commute

$$F \circ Gr_{w_r\lambda_r} \circ \ldots \circ Gr_{w_1\lambda_1} = \sum_{i=1}^{r} w_i\{m_{\lambda_i}, F\} \qquad \forall \lambda_i \in \pi_1(S_g), w_i \in \mathbb{R}^+$$

4. Relation $\{m_\lambda, s_\eta\} = \{s_\lambda, m_\eta\}\ \forall \lambda, \eta \in \pi_1(S_g)$.

5 Grafting and Dehn twists

Dehn twists:

(infinitesimal) Dehn twists along simple curves $\lambda \in \pi_1(S_g)$
$\Rightarrow$ transformation $D_{w\lambda} : ISO(2,1)^{2g} \to ISO(2,1)^{2g}$

1. infinitesimally generated via the Poisson bracket by observable $m_\lambda s_\lambda$

$$\frac{d}{dw}|_{w=0} F \circ D_{w\lambda} = \{m_\lambda s_\lambda, F\} \qquad \forall F \in C^\infty(ISO(2,1)^{2g})$$

2. Poisson isomorphism: $\{F \circ D_{w\lambda}, G \circ D_{w\lambda}\} = \{F, G\} \circ D_{w\lambda}$

3. Leaves constraint invariant and commutes with gauge transformations

4. Explicit formula for action on holonomy H_η, $\eta \in \pi_1(S_g)$:

- write curves as product in generators $a_i, b_i \in \pi_1(S_g)$

$$\lambda = x_r^{\alpha_r} \circ \ldots \circ x_1^{\alpha_1},\ \eta = y_s^{\beta_s} \circ \ldots \circ y_1^{\beta_1} \quad x_i, y_j \in \{a_1, ..., b_g\}, \alpha_i, \beta_j \in \{\pm 1\}$$

- intersection point between factors $x_{k+1}^{\alpha_{k+1}}$ and $x_k^{\alpha_k}$ on λ, $y_{l+1}^{\beta_{l+1}}$ and $y_l^{\alpha_l}$ on η

$$D_{w\lambda} : H_\eta \mapsto Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot H_\lambda^{\epsilon w} \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1}$$
$$H_\lambda^{\epsilon w} = e^{-\epsilon w (p_\lambda^a J_a + k_\lambda^a P_a)},\ H_\lambda = e^{-(p_\lambda^a J_a + k_\lambda^a P_a)} = X_r^{\alpha_r} \cdots X_1^{\alpha_1}$$

Grafting:

$Gr_{w m_\lambda \lambda}$:

$$\begin{aligned} H_\eta \mapsto & Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot (1, -w\epsilon \boldsymbol{p}_\lambda) \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1} \\ = & Y_s^{\beta_s} \cdots Y_{l+1}^{\beta_{l+1}} \cdot (X_k^{\alpha_k} \cdots X_1^{\alpha_1}) \cdot H_\lambda^{\epsilon w \theta} \cdot (X_1^{-\alpha_1} \cdots X_k^{-\alpha_k}) \cdot Y_l^{\beta_l} \cdots Y_1^{\beta_1} \end{aligned}$$

Grafting along λ =inf. Dehn twist along λ with formal parameter θ, $\theta^2 = 0$

$$\boxed{Gr_{w m_\lambda \lambda} = D_{\theta w \lambda}}$$

6 Outlook and Conclusions

Relation between geometrical construction of (2+1)-spacetimes via grafting and phase space and Poisson structure in the Chern-Simons formulation of (2+1)-dimensional gravity for $\Lambda = 0$, spacetimes $M \approx \mathbb{R} \times S_g$

- implementation of grafting along closed, simple $\lambda \in \pi_1(S_g)$ in the Chern-Simons formalism $\Rightarrow$ grafting transformation $Gr_{w\lambda}$ on Poisson manifold $(ISO(2,1)^{2g}, \Theta)$

- generated via the Poisson bracket by gauge invariant observable m_λ

- Poisson isomorphism, respects constraint, commutative

- general relation for Poisson brackets of mass and spin $\{m_\lambda, s_\eta\} = \{s_\lambda, m_\eta\}$

- can be viewed as (infinitesimal) Dehn twist along λ with formal parameter θ, $\theta^2 = 0$

$\Rightarrow$ **Physical interpretation of gauge invariant observables:**

m_λ: generates grafting: cuts spatial surface along λ and translates sides of the cut

$m_\lambda s_\lambda$: generates inf. Dehn twist: cuts spatial surface along λ and rotates sides of the cut

Open questions

- Other cases of cosmological constant $\Lambda > 0$, $\Lambda < 0$?

- Manifestation of Wick rotation [Benedetti, Bonsante] on phase space ?