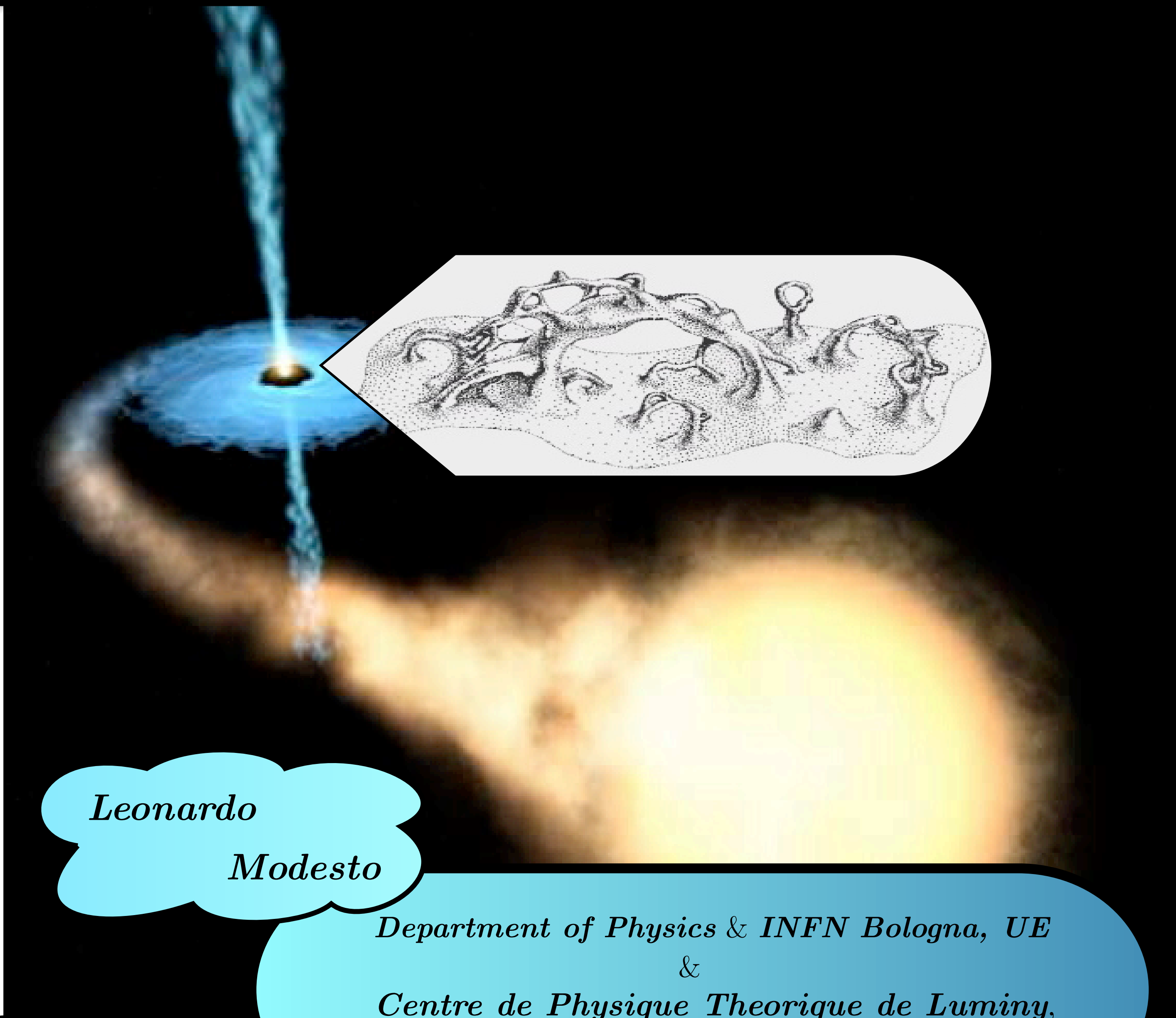
Disappearance of black Hole Singularity in Quantum Gravity



Leonardo Modesto

Department of Physics & INFN Bologna, UE
&
Centre de Physique Theorique de Luminy,
Universite de la Méditerranee, Marseille, EU

OUTLINE

Non singular black hole

The Kantowski-Sachs space-time

Loop quantum black hole

Quantum Gravitational Collapse

Schroedinger representation

***Physical system :** point particle on the real line R*
$\forall$complex number ζ we introduce an operator $W(\zeta)$, and consider the vector space W generated by them. (ζ dimensionless)
We introduce a product and an involution "$\star$" on W :

$$W(\zeta_1)\,W(\zeta_2) = e^{\frac{i}{2}\,Im(\zeta_1\bar{\zeta}_2)}\,W(\zeta_1+\zeta_2) \qquad [W(\zeta)]^\star = W(-\zeta)$$

This is the Weyl-Heisenberg $\star -$algebra.

In physics one introduce the length "d" and the splits operators $W(\zeta)$ by setting : $W(\zeta) = e^{\frac{i}{2}\,\lambda\,\mu}\,U(\lambda)\;V(\mu)$, $\quad \zeta = \lambda\,d + i\left(\dfrac{\mu}{d}\right)$

Thus, $U(\lambda) = W(\lambda\,d)$ and $V(\mu) = W(i\mu/d)$

The operators $U(\lambda)$ and $V(\mu)$ satisfy

$$[U(\lambda)]^\star = U(-\lambda), \quad [V(\lambda)]^\star = V(-\lambda)$$

$$U(\lambda_1)\,U(\lambda_2) = U(\lambda_1+\lambda_2), \quad V(\mu_1)\,V(\mu_2) = V(\mu_1+\mu_2)$$

$$U(\lambda)\,V(\mu) = e^{-i\lambda\mu}\;V(\mu)\,U(\lambda)$$

The Schroedinger representation

***The Stone - Von Neumann theorem :** every irreducible representation of W which is weakly continuous in the parameter ζ is unitarily equivalent to the standard Schrodinger representation, where the Hilbert space is the space $L^2(R, \underline{x})$ ($\underline{x}$ dimensionless).*

$W(\zeta)$ are represented via :

$$\widehat{W}(\zeta)\psi(\underline{x}) = e^{\frac{i}{2}\,\alpha\,\beta}\,e^{i\,\alpha\,\underline{x}}\,\psi(\underline{x}+\beta), \quad \zeta = \alpha + i\beta, \;\; (\alpha = \lambda d, \;\; \beta = \mu/d)$$

This is an irreducible representation of W.
The $\widehat{W}(\zeta)$ are all unitary (i.e., satisfy $[\widehat{W}]^\dagger = [\widehat{W}]^{-1}$).
The $\widehat{W}(\zeta)$ are weakly continuous in ζ (i.e., all matrix elements of $\widehat{W}(\zeta)$ are continuous in ζ).

In physics terms, the Hilbert space H_{Sch} is $L^2(R, dx)$ ($x = \underline{x}\,d$), and

$$\widehat{U}(\lambda)\psi(x) = e^{i\lambda x}\psi(x), \quad \widehat{V}(\mu)\psi(x) = \psi(x+\mu), \;\; \forall\;\; \psi(x) \in H_{Sch}$$

$\widehat{U}(\lambda)$ and $\widehat{V}(\mu)$ are weakly continuous in λ and μ, then exist self-adjoint operators $\hat{x}$ and $\hat{p}$ such that : $\widehat{U}(\lambda) := e^{i\lambda\hat{x}}$, $\;\widehat{V}(\mu) := e^{i\mu/\hbar\hat{p}}$

Polymer representation

This rep. is unitarily inequivalent to the Schroedinger rep.

Stone - Von Neumann theorem violation :

the operator $V(\mu) = e^{i\mu\hat{p}}$ *not weakly continuous in* μ.

Construction of H_{Poly}

- *We introduce a graph* γ : *it consists of a countable set* $\{x_i\}$, $x_i \in R$

Properties of the x_i *points* :

i) don't contain sequences with accumulation points in R,

ii) $\exists$ *constants* l_γ, ρ_γ *sach that the number* $n(I)$ *of points in any interval* I *of lenght* $l(I) \geqslant l_\gamma$ *is bounded by* $n(I) \leqslant \rho_\gamma l(I)$.

- *We denote by* Cyl_γ *the vector space of complex functions* $f(k)$:

$$f(k) = \sum_j f_j e^{-ix_j k}, \quad k \in R, \quad x_j \in R, \quad f_j \in C$$

$f(k)$: *cylindrical function with respect to the graph* γ.

- *Now we consider all possible graphs and denote by Cyl the infinite dimensional vector space of function on R which are cylindrical with respect to some graph* :

$$Cyl := \bigcup_\gamma Cyl_\gamma$$

Basis in Cyl : $e^{-i\,x_i\,k}$

uncountable basis labeled by arbitrary real numbers x_i.

Hermitian inner product on Cyl : $\langle\, e^{-i\,x_i\,k} | e^{-i\,x_j\,k} \rangle = \delta_{x_i, x_j}$

The Hilbert space H_{Poly} *is the Cauchy completion of Cyl.*

Weyl - Heisenberg algebra representation on H_{Poly} :

$$\hat{W}(\zeta) f(k) = [e^{\frac{i}{2}\lambda\mu}\, \hat{U}(\lambda)\, \hat{V}(\mu)]\, f(k)$$

$$\hat{U}(\lambda) f(k) = f(k-\lambda), \qquad \hat{V}(\mu) f(k) = e^{i\,\mu\,k} f(k)$$

Representation of W in terms of eigenkets of $\hat{U}(\lambda)$

We associate a ket $|x_j\rangle$ *to the basis elements* $e^{-ix_j k}$

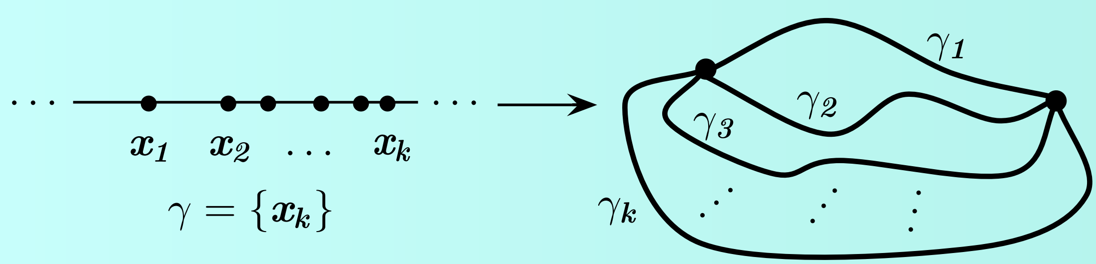
The action of $\hat{U}(\lambda)$ *and* $\hat{V}(\mu)$ *on the basis* $|x_j\rangle$ *is :*

$$\hat{U}(\lambda)|x_j\rangle = e^{i\lambda x_j}\,|x_j\rangle \quad , \quad \hat{V}(\mu)|x_j\rangle = |x_j - \mu\rangle$$

$\hat{U}(\lambda)$ *is weakly continuous in* λ *so :*

$$\hat{U}(\lambda) = e^{i\lambda\hat{x}} \quad \rightarrow \; \exists\; \hat{x} \; \rightarrow \quad \hat{x}|x_j\rangle = x_j|x_j\rangle$$

Polymer representation and Loop Quantum Gravity



$$k \;\rightarrow\; A \quad , \quad h_j(k) = e^{-ix_j k} \;\rightarrow\; h_\gamma = P\, e^{-\int_\gamma A}$$

Operator analogy

Holonomies : $\hat{V}(\mu) \;\rightarrow\; \hat{h}_\gamma[A]$, *Fluxes :* $\hat{x} \;\rightarrow\; \hat{E}[S]$

$$[\hat{x},\, \hat{V}(\mu)] = -\mu\,\hat{V}(\mu)$$

The Schwarzschild solution inside the horizon

$$ds^2 = -\frac{dT^2}{\left(\frac{2MG_N}{T} - 1\right)} + \left(\frac{2MG_N}{T} - 1\right) dr^2 + T^2(\sin^2\theta \, d\phi^2 + d\theta^2)$$

$$T \in]0, 2MG_N[\ , \ \ r \in]-\infty, +\infty[$$

We can eliminate the coefficient of dT^2 :

$$ds^2 = -d\tau^2 + \left(\frac{2MG_N}{T(\tau)} - 1\right) dr^2 + T(\tau)^2(\sin^2\theta \, d\phi^2 + d\theta^2)$$

$$\tau = -\sqrt{T(2MG_N - T)} + 2MG_N \arctan\left(\sqrt{\frac{T}{2MG_N - T}}\right)$$

This is the Kantowski-Sachs space-time $(R \times R \times S^2)$:

$$ds^2 = -N^2(t)dt^2 + a^2(t)dr^2 + b^2(t)(\sin^2\theta \, d\phi^2 + d\theta^2)$$

The classical theory

The Hamiltonian constraint :

$$H_L = |a| \, \dot{b}^2 + 2 \, \dot{a} \, \dot{b} \, b \, sgn(a) + |a|$$

A simplification of the problem

For $a^2(t) = \frac{2MG_N}{b(t)} - 1$, *we obtain* $H_L = \frac{R}{G_N}\left[\frac{\dot{b}^2}{\sqrt{\frac{2MG_N}{b} - 1}} - \sqrt{\frac{2MG_N}{b} - 1}\right]$

Near singularity limit $1 - \frac{b}{2MG_N} \sim 1$

$$H = \left(\frac{G_N \, p^2}{2 \, R} - \frac{R}{2 \, G_N}\right)\left[\frac{\sqrt{2MG_N}}{\sqrt{b}}\left(1 - \frac{b}{2MG_N}\right)^{\frac{1}{2}}\right] \rightarrow H = \left(\frac{G_N \, p^2}{2 \, R} - \frac{R}{2 \, G_N}\right)\frac{\sqrt{2MG_N}}{\sqrt{b}}$$

Volume operator :

$$V = 4\pi R\sqrt{2MG_N} \, b^{3/2}\sqrt{1 - \frac{b}{2MG_N}} \ \rightarrow \ V \sim 4\pi R\sqrt{2MG_N} \, b^{3/2} \equiv l_o \, b^{3/2}$$

Phase space and symplectic structure

The canonical pair is $(b \equiv x, p)$, *with Poisson bracket* $\{x, p\} = 1$

We are motivated by loop quantum gravity to use the fundamental variables :

$$\left(x,\ U_\gamma(p) \equiv e^{\frac{8\pi G_N \gamma}{L}\, i\, p}\right)$$

γ *is a real parameter and* L *fixes the unit of length,* $\gamma = l_P / L_{Phys}$

A straightforward calculation gives :

$$\{x, U_\gamma(p)\} = 8\pi G_N \frac{i\,\gamma}{L} U_\gamma(p),$$

$$U_\gamma^{-1}\{V^n, U_\gamma\} = l_0^n\, U_\gamma^{-1}\{|x|^{\frac{3n}{2}}, U_\gamma\} = i\, 8\pi G_N\, l_0^n\, \frac{\gamma}{L}\frac{3n}{2} sgn(x)|x|^{\frac{3n}{2}-1}$$

$$\text{For } n = 1/3, \quad \frac{sgn(x)}{\sqrt{|x|}} = -\frac{2L\, i}{(8\pi G_N) l_0^{\frac{1}{3}} \gamma}\, U_\gamma^{-1}\{V^{\frac{1}{3}}, U_\gamma\}$$

Quantum theory

Hilbert space : $L_2(\bar{R}_{Bohr}, d\mu_0)$

- $\bar{R}_{Bohr}$ *is the Bohr-compactification of* R
- $d\mu_0$ *is the Haar measure on* $\bar{R}_{Bohr}$

Operators : $(\hat{x},\ \hat{U}_\gamma)$

$\hat{U}_\gamma$ *is the analog of the classical operator* $U_\gamma = e^{i8\pi G_N \gamma p/L}$

$\hat{U}$ *is not weakly continuous in* γ

Basis states in the Hilbert space :

$$|\lambda\rangle \equiv |e^{i\lambda x/L}\rangle, \quad \langle\mu|\lambda\rangle = \delta_{\mu,\lambda}$$

Action of $\hat{x}$ *and* $\hat{U}_\gamma$ *on* $|\mu\rangle$:

$$\hat{x}|\mu\rangle = L\,\mu|\mu\rangle$$

$$\hat{U}_\gamma|\mu\rangle = |\mu - \gamma\rangle, \quad \left[\hat{x}, \hat{U}_\gamma\right] = -\gamma L \hat{U}_\gamma$$

$L = \sqrt{8\pi}\, l_p$

Volume operator and disappearance of singularity

The action of the volume operator on the basis states is :

$$\hat{V}|\mu\rangle = l_0 |x|^{\frac{3}{2}} |\mu\rangle = l_0 |L\mu|^{\frac{3}{2}} |\mu\rangle$$

The operator $\widehat{\frac{1}{|x|}}$ *and its spectrum :*

$$\widehat{\frac{1}{|x|}} = \frac{1}{2\pi l_p^2 l_0^{\frac{2}{3}}} \left(\hat{U}^{-1}\left[\hat{V}^{\frac{1}{3}}, \hat{U}\right]\right)^2 \quad , \quad \widehat{\frac{1}{|x|}}|\mu\rangle = \sqrt{\frac{2}{\pi l_p^2}} \left(|\mu|^{\frac{1}{2}} - |\mu - 1|^{\frac{1}{2}}\right)^2 |\mu\rangle$$

Spectrum of the curvature invariant operator :

$$R_{\mu\nu\rho\sigma} \widehat{R^{\mu\nu\rho\sigma}} |\mu\rangle = \frac{48\widehat{M^2 G_N^2}}{|x|^6}|\mu\rangle = \frac{384 M^2 G_N^2}{\pi^3 l_P^6} \left(|\mu|^{\frac{1}{2}} - |\mu - 1|^{\frac{1}{2}}\right)^{12} |\mu\rangle$$

The spectrum is singularity free for any eigenvalue μ

Hamiltonian Constraint

Using the classical expression $p^2 = \frac{L^2}{(8\pi G_N)^2} \lim_{\gamma \to 0} \left(\frac{2 - U_\gamma - U_\gamma^{-1}}{\gamma^2}\right)$ *we can write :*

$$\hat{H} = \frac{A_1}{l_0^{1/3}} \left[\hat{U}_\gamma + \hat{U}_\gamma^{-1} - (2 - A_2)\,1\right] sgn(x) \left(\hat{U}^{-1}\left[\hat{V}^{\frac{1}{3}}, \hat{U}\right]\right)$$

The solutions of the hamiltonian constraint are in the $C^\star$ *space that is the dual of the dense subspace* C *of the kinematical space* H.

A generic element of this space is $\langle\psi| = \sum \psi(\mu)\langle\mu|$.

The constraint equation $\hat{H}|\psi\rangle = 0$ *is now interpreted as an equation in the dual space* $\langle\psi|\hat{H}^\dagger$*; from this equation we obtain the*

DISCRETE DIFFERENCE EQUATION :

$$V_{\frac{1}{2}}(\mu + \gamma)\,\psi(\mu + \gamma) + V_{\frac{1}{2}}(\mu - \gamma)\,\psi(\mu - \gamma) - (2 - C')\; V_{\frac{1}{2}}(\mu)\,\psi(\mu) = 0$$

$$V_{\frac{1}{2}}(\mu) = -\left||\mu - \gamma|^{1/2} - |\mu|^{1/2}\right| \text{ for } \mu \neq 0 \text{ and } V_{\frac{1}{2}}(\mu) = |\gamma|^{1/2} \text{ for } \mu = 0$$

$$A_1 = \frac{L^3\; G_N}{(8\pi G_N)^{5/2} \gamma^3\, R\, l_0^{1/3}\, \hbar} \quad , \quad A_2 = \frac{8\pi R^2 \gamma^2}{l_P^2} \quad , \quad C = A_1 L^{1/2} \quad , \quad C' \equiv A_2$$

The Kantowski-Sachs Space-Time

Classical theory

$$ds^2 = -dt^2 + a^2(t)dr^2 + b^2(t)(\sin^2\theta d\phi^2 + d\theta^2)$$

Hamiltonian constraint and volume of the space section :

$$H_c = \frac{G_N\,|a|\,p_a^2}{2R\,b^2} - \frac{G_N\,p_a p_b\,sgn(a)}{Rb} - \frac{R}{2G_N}|a|$$

$$V = \int dr\,d\phi\,d\theta\,h^{1/2} = 4\pi R|a|b^2$$

Canonical pairs : $(a \equiv x_a, p_a)$ *and* $(b \equiv x_b, p_b)$

Poisson brackets : $\{x_a, p_a\} = 1 \quad , \quad \{x_b, p_b\} = 1$

As in Loop Quantum Gravity we use the fundamental variables :

$$\left(x_a,\, U_{\gamma_a}(p) \equiv exp\Big(\frac{8\pi G_N\gamma_a}{L_a^2}\, i\, p_a\Big)\right)\ ,\ \left(x_b,\, U_{\gamma_b}(p) \equiv exp\Big(\frac{8\pi G_N\gamma_b}{L_b}\, i\, p_b\Big)\right)$$

We have also that :

$$\{x_a, U_{\gamma_a}(p_a)\} = 8\pi G_N \frac{i\,\gamma_a}{L_a^2} U_{\gamma_a}(p_a)\ ,\ \{x_b, U_{\gamma_b}(p_b)\} = 8\pi G_N \frac{i\,\gamma_b}{L_b} U_{\gamma_b}(p_b)$$

$$U_{\gamma_a}^{-1}\{V^m, U_{\gamma_a}\} = (4\pi R|x_b|^2)^m\, m\,|x_a|^{m-1}\, i\gamma_a \frac{8\pi G_N}{L_a^2} sgn(x_a)$$

$$U_{\gamma_b}^{-1}\{V^n, U_{\gamma_b}\} = (4\pi R|x_a|)^n\, 2n\,|x_b|^{2n-1}\, i\gamma_b \frac{8\pi G_N}{L_b} sgn(x_b)$$

From those relations we construct the following quantities :

$$\frac{|x_b|^{2/3}}{|x_a|^{2/3}} = -\frac{3\,i\,L_a^2}{(4\pi R)^{\frac{1}{3}}\,8\pi G_N\gamma_a}\, U_{\gamma_a}^{-1}\{V^{\frac{1}{3}}, U_{\gamma_a}\}\, sgn(x_a)$$

$$\frac{|x_a|^{1/4}}{|x_b|^{1/2}} = -\frac{2\,i\,L_b}{(4\pi R)^{\frac{1}{4}}\,8\pi G_N\gamma_b}\, U_{\gamma_b}^{-1}\{V^{\frac{1}{4}}, U_{\gamma_b}\}\, sgn(x_b)$$

$$\sqrt{|x_a|} = -\frac{i\,L_b}{(4\pi R)^{\frac{1}{2}}\,8\pi G_N\gamma_b}\, U_{\gamma_b}^{-1}\{V^{\frac{1}{2}}, U_{\gamma_b}\}\, sgn(x_b)$$

$$\frac{|x_a|^{1/3}}{|x_b|^{1/3}} = -\frac{3\,i\,L_b}{2\,(4\pi R)^{\frac{1}{3}}\,8\pi G_N\gamma_b}\, U_{\gamma_b}^{-1}\{V^{\frac{1}{3}}, U_{\gamma_b}\}\, sgn(x_b)$$

Quantum Theory

$$\textit{Hilbert space} : \; L_2(\bar{R}^2_{Bohr}, d\mu_0)$$

Basis states in the complete Hilbert space :

$$|\lambda_a\rangle \otimes |\lambda_b\rangle \equiv |e^{i\lambda_a x_a}\rangle \otimes |e^{i\lambda_b x_b/L_b}\rangle \quad , \quad \langle \mu_a|\lambda_a\rangle = \delta_{\mu_a,\lambda_a} \quad , \quad \langle \mu_b|\lambda_b\rangle = \delta_{\mu_b,\lambda_b}$$

$$\hat{x}_a|\mu_a\rangle = \mu_a|\mu_a\rangle \quad , \quad \hat{x}_b|\mu_b\rangle = L_b\mu_b|\mu_b\rangle$$

The quantum theory is defined by :

$$\left(x_a, \hat{U}_{\gamma_a}\right) \quad , \quad \left(x_b, \hat{U}_{\gamma_b}\right)$$

$$\hat{U}_{\gamma_a}|\mu_a\rangle = |\mu_a - \gamma_a\rangle \quad , \quad \hat{U}_{\gamma_b}|\mu_b\rangle = |\mu_b - \gamma_b\rangle$$

$$\left[\hat{x}_a, \hat{U}_{\gamma_a}\right] = -\gamma_a \hat{U}_{\gamma_a} \quad , \quad \left[\hat{x}_b, \hat{U}_{\gamma_b}\right] = -\gamma_b\, L_b\, \hat{U}_{\gamma_b}$$

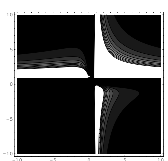
$$L_b = \sqrt{8\pi}\, l_p$$

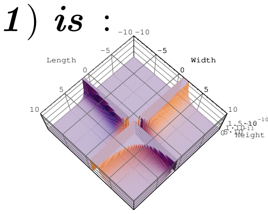
The Volume Inverse Operator and Singularity Resolution

$$\hat{V}\,|\mu,\nu\rangle = 4\pi R\,|\hat{x}_a|\,|\hat{x}_b|^2|\mu,\nu\rangle = 4\pi R L_b^2\,|\mu|\,|\nu|^2|\mu,\nu\rangle$$

The spectrum of $\widehat{1/V}$ ***(for*** $\gamma_a = \gamma_b = 1$***) is :***

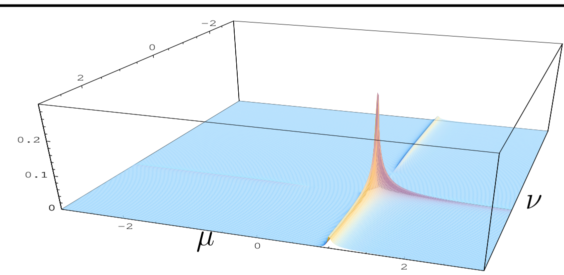
$$\widehat{\frac{1}{det(E)}} = \left(\widehat{\frac{|x_a|}{|x_b|^2}}\right)^3_{\gamma_b} \left(\widehat{\frac{|x_b|^2}{|x_a|^2}}\right)^3_{\gamma_a} \left(\widehat{\frac{|x_a|}{|x_b|}}\right)^2_{\gamma_b}$$





$$\widehat{\frac{1}{det(E)}}|\mu,\nu\rangle = \frac{2^6\, 3^{15}}{L^2}\,|\mu|^5\,|\nu|^6\,[|\nu-1|^{\frac{1}{2}} - |\nu|^{\frac{1}{2}}]^{12}\,||\mu-1|^{\frac{1}{3}} - |\mu|^{\frac{1}{3}}|^9\,[|\nu-1|^{\frac{2}{3}} - |\nu|^{\frac{2}{3}}]^6$$

The operator $1/|x_b| \;\rightarrow R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} \sim \frac{1}{x_b^6}$

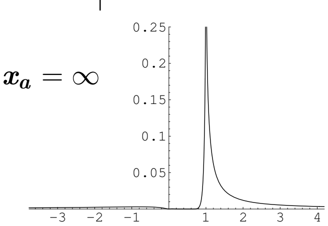


The spectrum :

$$\widehat{\frac{1}{|x_b|}}|\mu,\nu\rangle = \frac{2\, 3^6}{L}\,|\mu|^2\left||\mu-1|^{\frac{1}{3}} - |\mu|^{\frac{1}{3}}\right|^3|\nu|^2\left[|\nu-1|^{\frac{1}{2}} - |\nu|^{\frac{1}{2}}\right]^4\left||\nu-1|^{\frac{2}{3}} - |\nu|^{\frac{2}{3}}\right|^3|\mu,\nu\rangle$$

From the Schwarzschild solution the singular point is in $x_b = 0$ $x_a = \infty$

$$\widehat{\frac{1}{|x_b|}}|\mu,\nu\rangle \rightarrow \frac{2\, 3^3}{L}\,|\nu|^2\left[|\nu-1|^{\frac{1}{2}} - |\nu|^{\frac{1}{2}}\right]^4\left||\nu-1|^{\frac{2}{3}} - |\nu|^{\frac{2}{3}}\right|^3|\mu,\nu\rangle$$



Hamiltonian Constraint

Classical Hamiltonian constraint

$$H_c = \frac{G_N\, p_a^2}{2R}\, \frac{|x_a|}{x_b^2} - \frac{G_N\, p_a\, p_b}{R}\, \frac{sgn(x_b)\, sgn(x_a)}{|x_b|} - \frac{R}{2G_N}\, |x_a|$$

Quantum Hamiltonian constraint

$$\begin{aligned}\hat{H} = & \frac{1}{32\pi^2\, G_N R^2 \gamma_a^2 \gamma_b^4} \left[2 - \hat{U}_a - \hat{U}_a^{-1}\right] \left(\hat{U}_b^{\,-1} \left[\hat{V}^{\frac{1}{4}}, \hat{U}_b\right]\right)^4 \\ & + \frac{3^6}{2^{11}\pi^5 R^4 L^4\, G_N \gamma_a^7 \gamma_b^5} \left[\left(\frac{\hat{U}_a + \hat{U}_b - \hat{U}_a\, \hat{U}_b - 1}{2}\right) + h.c.\right] \cdot \\ & \cdot \left(\hat{U}_b^{\,-1} \left[\hat{V}^{\frac{1}{4}}, \hat{U}_b\right]\right)^4 \left(\hat{U}_a^{\,-1} \left[\hat{V}^{\frac{1}{3}}, \hat{U}_a\right]\right)^3 \left(\hat{U}_b^{\,-1} \left[\hat{V}^{\frac{1}{3}}, \hat{U}_b\right]\right)^3 \\ & - \frac{1}{8\pi\, G_N L^2 \gamma_b^2} \left(\hat{U}_b^{\,-1} \left[\hat{V}^{\frac{1}{2}}, \hat{U}_b\right]\right)^2\end{aligned}$$

Solutions of the Hamiltonian constraint

The solutions of the Hamiltonian constraint are in the $C^\star$ space;
this is the dual of the dense subspace C of the kinematical space H.
A generic element of this space is $\langle\psi| = \sum_{\mu,\nu} \psi(\mu,\nu)\langle\mu,\nu|$,
and the action of $\hat{H}$ on this state is $\langle\psi|\hat{H}^\dagger$.
From this equation we can derive a relation for the coefficients $\psi(\mu,\nu)$:

$$\begin{aligned}&[2\alpha(\mu,\nu) - 2\beta(\mu,\nu) + \gamma(\mu,\nu)]\,\psi(\mu,\nu) - [\alpha(\mu+\gamma_a,\nu) - \beta(\mu+\gamma_a,\nu)]\,\psi(\mu+\gamma_a,\nu) \\ &-[\alpha(\mu-\gamma_a,\nu) + \beta(\mu-\gamma_a,\nu)]\,\psi(\mu-\gamma_a,\nu) + \beta(\mu,\nu+\gamma_b)\,\psi(\mu,\nu+\gamma_b) \\ &-\beta(\mu,\nu-\gamma_b)\,\psi(\mu,\nu-\gamma_b) + \beta(\mu+\gamma_a,\nu+\gamma_b)\,\psi(\mu+\gamma_a,\nu+\gamma_b) \\ &-\beta(\mu-\gamma_a,\nu-\gamma_b)\,\psi(\mu-\gamma_a,\nu-\gamma_b) = 0\end{aligned}$$

This is the DISCRETE DIFFERENCE EQUATION for
the KANTOWSKI - SACHS space-time

Classical Gravitational Collapse

Space-time inside the Horizon

Outside the matter :

$$ds^2 = -\tilde{N}(t)dt^2 + \tilde{a}^2(t)dr^2 + b^2(t)(\sin^2\theta d\phi^2 + d\theta^2)$$

Inside the matter :

$$ds^2 = -N(t)dt^2 + a^2(t)\big[d\chi^2 + \sin^2\chi(\sin^2\theta d\phi^2 + d\theta^2)\big].$$

Volume operators inside and outside the matter :

$$V_{in} = \int_0^{\chi_0} d\chi \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, h_{in}^{1/2} = 2\pi\big(\chi_0 - \sin(\chi_0)\cos(\chi_0)\big)\,|a|^3 \equiv V(\chi_0)\,|a|^3$$

$$V_{out} = \int_0^{R} dr \int_0^{2\pi} d\phi \int_0^{\pi} d\theta\, h_{out}^{1/2} = 4\pi R|\tilde{a}|b^2$$

The Hamiltonian constraints are :

$$H_{out} = \frac{G_N\,|\tilde{a}|\,p_{\tilde{a}}^2}{2R\,b^2} - \frac{G_N\,p_{\tilde{a}}p_b\,sgn(\tilde{a})}{Rb} - \frac{R}{2G_N}|\tilde{a}| \quad , \quad outside\ matter$$

$$H_{in} = -\Big(\frac{p_a^2}{8|a|} + 2|a|\Big) + \frac{16\pi G_N}{3}H_\phi(a) \quad , \quad inside\ matter$$

Inside the Matter

Gravity Sector

$$Fundamental\ variables:\ (x_a, U_{\gamma_a}) \ ,\ U_{\gamma_a}(p_a) \equiv exp\Big(\frac{i\gamma_a}{L_a}\,p_a\Big)$$

$$\{x_a, U_{\gamma_a}\} = i\,\frac{8\pi G_N\gamma_a}{L_a}\,U_{\gamma_a} \quad , \quad U_{\gamma_a}^{-1}\{V_{in}^n, U_{\gamma_a}\} = i\,\frac{24\pi G_N\gamma_a}{L_a}\,n\,|x_a|^{3n-1}\,sgn(x_a)\,V^n(\chi_0)$$

$$\frac{sgn(x_a)}{\sqrt{|x_a|}} = -\frac{2L_a i}{8\pi G_N\gamma_a V^{1/6}(\chi_0)}\,U_{\gamma_a}^{-1}\{V_{in}^{\frac{1}{6}}, U_{\gamma_a}\}$$

Matter sector : DUST MATTER

$$H_\phi = p_\phi \ ,\ Canonical\ pair:\ (\phi, p_\phi)\ ,\ \{\phi, p_\phi\} = 1$$

$$Fundamental\ variables:\ (\phi,\ U_{\gamma_\phi}) \ ,\ U_{\gamma_\phi}(p_\phi) \equiv exp\Big(\frac{i\gamma_\phi\,8\pi G_N}{L_\phi}\,p_\phi\Big)$$

$$\{x_\phi, U_{\gamma_\phi}\} = i\,\frac{\gamma_\phi\,8\pi G_N}{L_\phi}\,U_{\gamma_\phi}$$

Quantu theory inside the matter

Hilbert space inside the matter $L_2(\hat{R}^2_{Bhor}, d\mu_0)$

$$|\lambda_a\rangle \otimes |\lambda_\phi\rangle \equiv |e^{i\lambda_a x_a/L_a}\rangle \otimes |e^{i\lambda_\phi x_\phi/L_\phi}\rangle \quad , \quad \langle\mu_a|\lambda_a\rangle = \delta_{\mu_a,\lambda_a} \quad , \quad \langle\mu_\phi|\lambda_\phi\rangle = \delta_{\mu_\phi,\lambda_\phi}$$

The quantum theory is defined by :

$$\left(\hat{x}_a, \hat{U}_{\gamma_a}\right) \quad , \quad \left(\hat{x}_\phi, \hat{U}_{\gamma_\phi}\right)$$

$$\hat{U}_{\gamma_a}|\mu_a\rangle = |\mu_a - \gamma_a\rangle \quad , \quad \hat{U}_{\gamma_\phi}|\mu_\phi\rangle = |\mu_\phi - \gamma_\phi\rangle$$

$$\left[\hat{x}_a, \hat{U}_{\gamma_a}\right] = -\gamma_a L_a \hat{U}_{\gamma_a} \quad , \quad \left[\hat{x}_\phi, \hat{U}_{\gamma_\phi}\right] = -\gamma_\phi L_\phi \hat{U}_{\gamma_\phi}$$

$$L_a = L_\phi = \sqrt{8\pi} l_p$$

Singularity resolution in the quantum theory

Spectrum of the volume operator : $\widehat{V}_{in} = V(\chi_0)\widehat{|x_a|}^3$

$$\widehat{V}_{in}|\mu_a, \mu_\phi\rangle = V(\chi_0)|\mu_a|^3|\mu_a, \mu_\phi\rangle$$

SPECTRUM OF THE INVERSE VOLUME OPERATOR :

$$\widehat{\frac{1}{V}}|\mu_a, \mu_\phi\rangle \sim \left(\widehat{\frac{1}{|x_a|}}\right)^3 |\mu_a, \mu_\phi\rangle = \left(\frac{2}{\pi l_p^2}\right)^{3/2} \left(|\mu_a|^{\frac{1}{2}} - |\mu_a - 1|^{\frac{1}{2}}\right)^6 |\mu_a, \mu_\phi\rangle$$

this spectrum is bounded from below

Quantum theory outside the matter

Outside the matter we have the Kantowski-Sachs space time and the quantization was developed in the context Schwarzschild singularity

Hamiltonian constraint inside the matter

$$H_{in} = -\left(\frac{p_a^2}{8}\frac{1}{|x_a|} + \frac{2}{V^{1/3}(\chi_0)}V^{1/3}\right) + \frac{16\pi G_N}{3}H_\phi(a)$$

The solution of the Hamiltonian constraint is in the dual space

of elements $\langle\psi| = \sum_{\mu_a,\mu_\phi}\psi(\mu_a,\mu_\phi)\langle\mu_a,\mu_\phi|.$

The equation for the coefficients $\psi(\mu_a,\mu_\phi)$ ***is*** :

$$\alpha(\mu_a)\psi(\mu_a,\mu_\phi) + \beta(\mu_a+\gamma_a)\psi(\mu_a+\gamma_a,\mu_\phi) + \beta(\mu_a-\gamma_a)\psi(\mu_a-\gamma_a,\mu_\phi) = -\frac{16\pi G_N}{3}\hat{H}_\phi(a)\psi(\mu_a,\mu_\phi)$$

$$\alpha(\mu_a) = -\frac{\sqrt{8\pi}\,l_P}{\gamma_a^4}\left(|\mu_a-\gamma_a|^{\frac{1}{2}} - |\mu_a|^{\frac{1}{2}}\right)^2 - 2\sqrt{8\pi}\,l_P\,|\mu_a| \qquad \beta(\mu_a) = \frac{\sqrt{8\pi}l_P}{2\gamma_a^4}\left(|\mu_a-\gamma_a|^{\frac{1}{2}} - |\mu_a|^{\frac{1}{2}}\right)^2$$

Boundary condition and time arrow

The operators area of S^2 ***inside and outside the matter are*** :

$$\widehat{A}_{in} = 4\pi\widehat{|x_a|}^2 sin(\chi_0) \quad , \quad \widehat{A}_{out} = 4\pi\widehat{|x_b|}^2$$

Spectrum of the two operators :

$$\widehat{A}_{in}|\mu_a,\mu_\phi\rangle = 4\pi\widehat{|x_a|}^2 sin^2(\chi_0)\,|\mu_a,\mu_\phi\rangle = 4\pi|\mu_a|^2 sin^2(\chi_0)\,|\mu_a,\mu_\phi\rangle$$

$$\widehat{A}_{out}|\mu_{\tilde{a}},\mu_b\rangle = 4\pi\widehat{|x_b|}^2\,|\mu_{\tilde{a}},\mu_b\rangle = 4\pi|\mu_b|^2\,|\mu_{\tilde{a}},\mu_b\rangle$$

At this point we identify the inside and outside spectrum :

$$|\mu_a|^2 sin^2(\chi_0) = |\mu_b|^2$$

If in the region outside the matter we assume that ν_b ***is the evolution parameter for our wave function, than the boundary condition (area matching) implies that the evolution parameter inside is*** μ_a ***given by*** $\mu_b = \mu_a \sin\chi_0$.

The boundary condition of the wave function on S^2 ***implies*** :

$$\psi_{IN}(a,\phi=0) = \psi_{OUT}\left(b, \tilde{a} = \frac{b}{\sin(\chi_0)}\right) \to \psi_{IN}(\mu_a,\phi=0) = \psi_{OUT}(\mu_a\sin(\chi_0),\mu_a)$$

this is the isotropy condition on the boundary.

Loop quantum black hole

Invariant 1-form connection $A_{[1]}$:

$$A_{[1]} = A_r(t)\,\tau_3\,dr + (A_1(t)\,\tau_1 + A_2(t)\tau_2)\,d\theta + (A_1(t)\,\tau_2 - A_2(t)\tau_1)\sin\theta\,d\phi + \tau_3\,\cos\theta\,d\phi$$

Invariant densitized triad :

$$E_{[1]} = E^r(t)\,\tau_3\,\sin\theta\,\frac{\partial}{\partial r} + (E^1(t)\,\tau_1 + E^2(t)\,\tau_2)\,\sin\theta\,\frac{\partial}{\partial\theta} + (E^1(t)\,\tau_2 - E^2(t)\tau_1)\frac{\partial}{\partial\phi}$$

Gauss constraint and Hamiltonian constrains :

$$G \sim A_1 E^2 - A_2 E^1$$

$$H_E = \frac{sgn[det(E_{[1]})]}{\sqrt{|E^r|[(E^1)^2 + (E^2)^2]}}\left[2A_r E^r(A_1 E^1 + A_2 E^2) + \Big((A_1)^2 + (A_2)^2 - 1\Big)[(E^1)^2 + (E^2)^2]\right]$$

For the Kantowski-Sachs space-time we fix the gauge
$E^2 = E^1$ ***and so*** $A_2 = A_1$

The Hamiltonian constraint becomes :

$$H_E = \frac{sgn(E)}{\sqrt{|E|}|E^1|}\left[2AEA_1E^1 + (2(A_1)^2 - 1)(E^1)^2\right]$$

Volume of the spatial section : $V = \int dr\,d\phi\,d\theta\,\sqrt{q} = 4\pi\sqrt{2}R\sqrt{|E|}|E^1|$

Background triad and co-triad :

$${}^o e^a_I = diag(1, 1, \sin^{-1}\theta) \qquad {}^o\omega^I_a = diag(1, 1, \sin\theta)$$

Holonomies

$$h_1 = \exp[A\mu_0 l_P\,\tau_3] \qquad h_2 = \exp[A_1\mu_0\,(\tau_2 + \tau_1)] \qquad h_3 = \exp[A_1\mu_0\,(\tau_2 - \tau_1)]$$

Hamiltonian constraint in terms of holonomies :

$$H_E = -\frac{8\pi}{\mu_0^3}\sum_{IJK}\epsilon^{IJK}\,Tr\left[h_I h_J h_I^{-1} h_J^{-1} h_{[IJ]}\,h_K^{-1}\{h_K, V\}\right]$$

$h_{[IJ]} = \exp(-\mu_0^2\,C_{IJ}\,\tau_3)$, $C_{IJ} = \delta_{2I}\delta_{3J} - \delta_{3_I}\delta_{2J}$ $\qquad \mu_0 = l_P/L_{Phys}$

Classical phase space :

Canonical pairs : (A, E) *and* (A_1, E^1)

Symplectic structure : $\{A, E\} = \frac{\kappa}{l_P}$, $\{A_1, E^1\} = \frac{\kappa}{4 l_P}$

Quantm theory

Hilbert space : $H_E \otimes H_{E^1} \sim L^2(R^2_{Bohr})$

Basis in the Hilbert space :

$$|\mu_E, \mu_{E^1}\rangle \equiv |\mu_E\rangle \otimes |\mu_{E^1}\rangle \to \langle A|\mu_E\rangle \otimes \langle A_1|\mu_{E^1}\rangle = e^{\frac{i\mu_E l_P A}{2}} \otimes e^{\frac{i\mu_{E^1} A_1}{\sqrt{2}}}$$

$$\langle \mu_E, \mu_{E^1}|\nu_E, \nu_{E^1}\rangle = \delta_{\mu_E,\nu_E}\,\delta_{\mu_{E^1},\nu_{E^1}}$$

Representation of the momentum operators :

$$\hat{E} \to -i l_P \frac{d}{dA} \quad , \quad \hat{E}^1 \to -i \frac{l_P}{4} \frac{d}{dA_1}$$

$$\hat{E}|\mu_E, \mu_{E^1}\rangle = \frac{\mu_E\, l_P^2}{2}|\mu_E, \mu_{E^1}\rangle \quad , \quad \hat{E}^1|\mu_E, \mu_{E^1}\rangle = \frac{\mu_{E^1}\, l_P}{4\sqrt{2}}|\mu_E, \mu_{E^1}\rangle$$

Inverse volume operator

$$\frac{\widehat{sgn(E)}}{\sqrt{\bar{V}}} = \frac{512\, i}{3\, l_P^4 \mu_0^3}\,\epsilon_{ijk} \sum_{IJK} \epsilon^{IJK}\; Tr\left[\tau^i \hat{h}_I^{-1}[\hat{h}_I, \hat{\bar{V}}^{\frac{1}{2}}]\right]\; Tr\left[\tau^j \hat{h}_J^{-1}[\hat{h}_J, \hat{\bar{V}}^{\frac{1}{2}}]\right]\; Tr\left[\tau^k \hat{h}_K^{-1}[\hat{h}_K, \hat{\bar{V}}^{\frac{1}{2}}]\right]$$

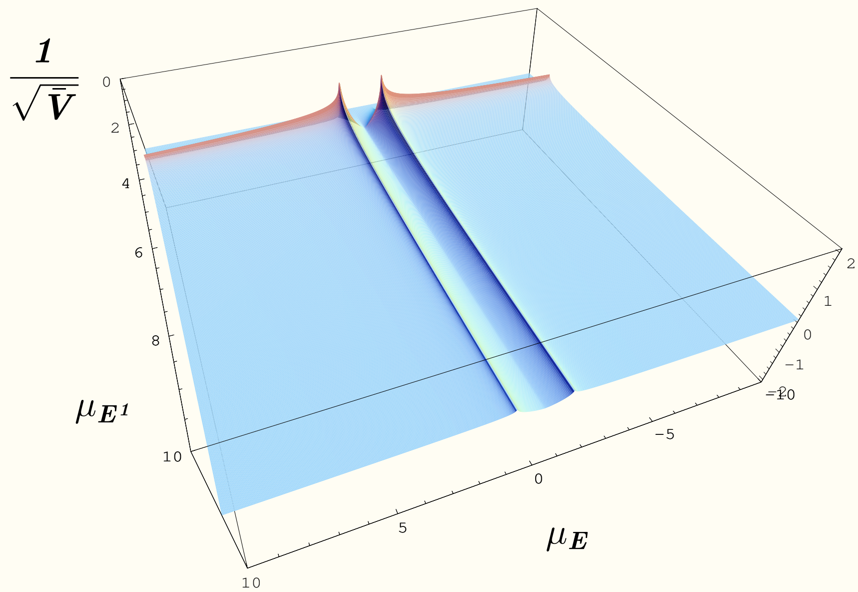
Spectrum of $\widehat{\bar{V}}$ *and* $\widehat{1/\bar{V}}$

$$\hat{V}|\mu_E, \mu_{E^1}\rangle = \frac{4\pi\, l_P^3}{\sqrt{2}}\,\sqrt{|\mu_E|}\,|\mu_{E^1}|\,|\mu_E, \mu_{E^1}\rangle$$

$$\frac{\widehat{sgn(E)}}{\sqrt{\bar{V}}}|\mu_E, \mu_{E^1}\rangle = \frac{8}{\sqrt{2}\, l_P \mu_0^3}\,|\mu_E|^{\frac{1}{2}}\,|\mu_{E^1}|^{\frac{1}{2}}$$

$$\left(|\mu_E + \mu_0|^{\frac{1}{4}} - |\mu_E - \mu_0|^{\frac{1}{4}}\right)\left(|\mu_{E^1} + \mu_0|^{\frac{1}{2}} - |\mu_{E^1} - \mu_0|^{\frac{1}{2}}\right)^2 |\mu_E, \mu_{E^1}\rangle$$

Plot of the $1/\sqrt{\bar{V}}$ operator spectrum



μ_E and μ_{E^1} are considered continuous variables

Hamiltonian constraint

The solutions of the Hamiltonian constraint are in $C^\star$ dual of the dense subspace C of the kinematical space H_{kin}. A generic element of this space is: $\langle\psi| = \sum_{\mu_E,\mu_{E^1}} \psi(\mu_E,\mu_{E^1})\langle\mu_E,\mu_{E^1}|$.

The constraint equation $\hat{H}_E|\psi\rangle = 0$ gives a relation for the coefficients $\psi(\mu_E,\nu_{E^1})$:

$$
\begin{aligned}
&-\alpha(\mu_E-2\mu_0,\mu_{E^1}-2\mu_0)\,\psi(\mu_E-2\mu_0,\mu_{E^1}-2\mu_0)\\
&+\alpha(\mu_E+2\mu_0,\mu_{E^1}-2\mu_0)\,\psi(\mu_E+2\mu_0,\mu_{E^1}-2\mu_0)\\
&+\alpha(\mu_E-2\mu_0,\mu_{E^1}+2\mu_0)\,\psi(\mu_E-2\mu_0,\mu_{E^1}+2\mu_0)\\
&-\alpha(\mu_E+2\mu_0,\mu_{E^1}+2\mu_0)\,\psi(\mu_E+2\mu_0,\mu_{E^1}+2\mu_0)\\
&+\Big(\frac{\sin(\mu_0^2/2)-\cos(\mu_0^2/2)}{2}\Big)\Big(\beta(\mu_E,\mu_{E^1}-4\mu_0)\,\psi(\mu_E,\mu_{E^1}-4\mu_0)\\
&\quad-\beta(\mu_E,\mu_{E^1})\,\psi(\mu_E,\mu_{E^1})+\beta(\mu_E,\mu_{E^1}+4\mu_0)\,\psi(\mu_E,\mu_{E^1}+4\mu_0)\Big)\\
&-\sin(\mu_0^2/2)\Big(\beta(\mu_E,\mu_{E^1}-2\mu_0)\,\psi(\mu_E,\mu_{E^1}-2\mu_0)\\
&\quad+\beta(\mu_E,\mu_{E^1}+2\mu_0)\,\psi(\mu_E,\mu_{E^1}+2\mu_0)\Big)=0
\end{aligned}
$$

$$\alpha(\mu_E,\mu_{E^1}) \equiv |\mu_E|^{\frac{1}{2}}\,(|\mu_{E^1}+\mu_0|-|\mu_{E^1}-\mu_0|)$$

$$\beta(\mu_E,\mu_{E^1}) \equiv |\mu_{E^1}|\,\Big(|\mu_E+\mu_0|^{\frac{1}{2}}-|\mu_E-\mu_0|^{\frac{1}{2}}\Big)$$

CONCLUSIONS

The classical black hole singularity near $r = b(t) \sim 0$ disappears from the quantum theory. Classical divergent quantities are bounded in the quantum theory.

- ***Curvature invariant:***

$$R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma} = \frac{48M^2\, G_N^2}{b(t)^6} \quad \to \quad \widehat{R_{\mu\nu\rho\sigma}\, R^{\mu\nu\rho\sigma}}|\psi\rangle = \frac{\widehat{48M^2\, G_N^2}}{b^6}|\psi\rangle$$

is bounded for the Kantowski-Sachs model.

- ***The inverse volume operator $1/\sqrt{V}$ is bounded.***

The quantum Hamiltonian constraint gives a discrete difference equation for the coefficients of the physical states and we can evolve across the classical singular point.

... INSIDE ... ACROSS ... AND BEYOND ...