STATES ON TIMELIKE HYPERSURFACES: A STEP TOWARDS GENERAL COVARIANCE IN QUANTUM FIELD THEORY

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hep-th/0509123
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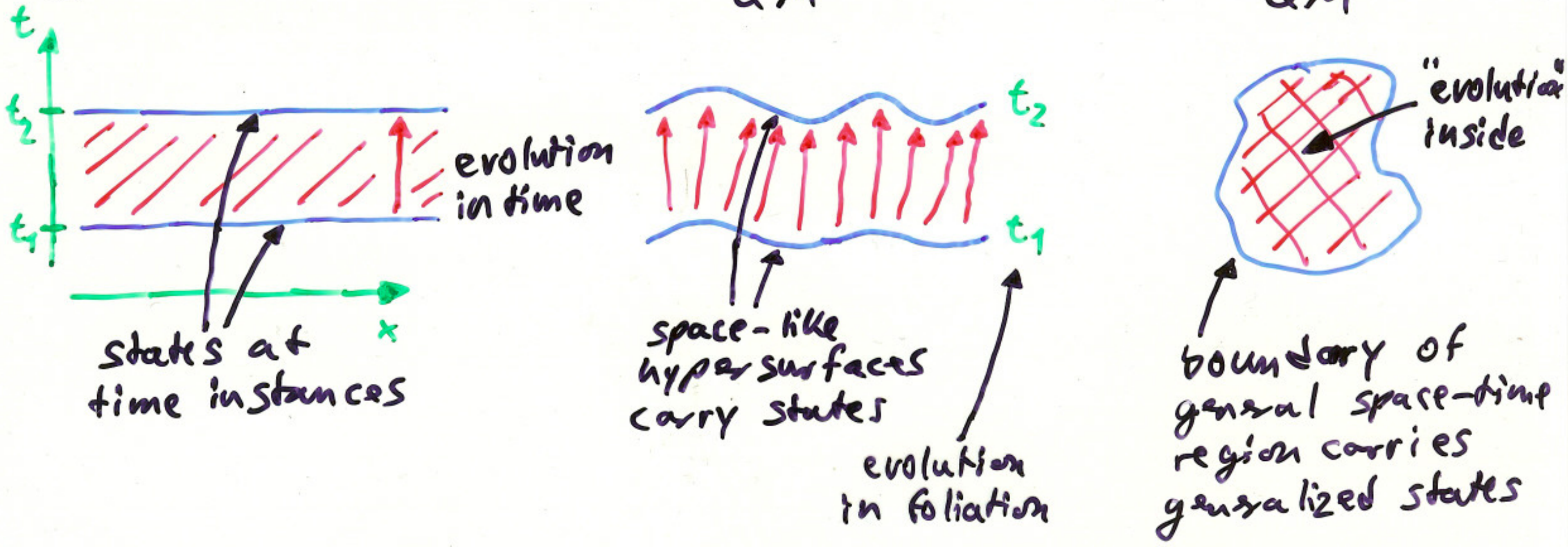
THE GENERAL BOUNDARY FORMULATION

on axiomatic level

QM + TQFT = general boundary QM

standard QM | curved space-time QM | general boundary QM



- ▷ associate generalized state spaces to boundaries of regions of space-time
- ▷ associate "transition" amplitudes to regions themselves

features

- ▷ avoid interpretational problems of combining GR with standard QM (notably problem of time)
- ▷ preserve standard QM where applicable
- ▷ local description of measurement process
- ▷ distinction between "in" and "out" states and between "preparation" and "observation" disappears
- ▷ interpretation: "collapse of wavefunction" is delocalized in time

CORE AXIOMS

M spacetime region

Σ oriented hypersurface

$\bar{\Sigma}$ hypersurface with opposite orientation

(T1) for any Σ have complex state space $\mathcal{H}_\Sigma$

(T1b) for any Σ have antilinear conjugation $\iota_\Sigma : \mathcal{H}_\Sigma \to \mathcal{H}_{\bar{\Sigma}}$
$\iota_{\bar{\Sigma}} \circ \iota_\Sigma = \mathrm{id}_\Sigma$

(T2) suppose $\Sigma = \Sigma_1 \cup \Sigma_2$ disconnected
$\Rightarrow \mathcal{H}_\Sigma = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$

(T2b) as (T2) $\Rightarrow$ $\iota_\Sigma = \iota_{\Sigma_1} \otimes \iota_{\Sigma_2}$

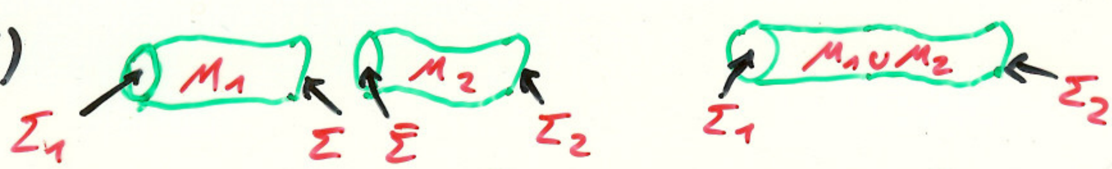
(T3) for any Σ have bilin. form $(\cdot,\cdot)_\Sigma : \mathcal{H}_{\bar{\Sigma}} \otimes \mathcal{H}_\Sigma \to \mathbb{C}$
s.t. $\langle \cdot,\cdot \rangle_\Sigma := (\iota_\Sigma(\cdot),\cdot)_\Sigma$ defines inner product
making $\mathcal{H}_\Sigma$ into Hilbert space

(T3b) bilin. form (T3) is compatible with tensor prod. (T2)

(T4) for any M with boundary Σ have linear
amplitude map $\rho_M : \mathcal{H}_\Sigma \to \mathbb{C}$

(T4b) suppose M has disconnected boundary $\Sigma = \Sigma_1 \cup \Sigma_2$
and amplitude induces isom. $\tilde{\rho}_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\bar{\Sigma}_2}$
then $\tilde{\rho}_M$ must be unitary

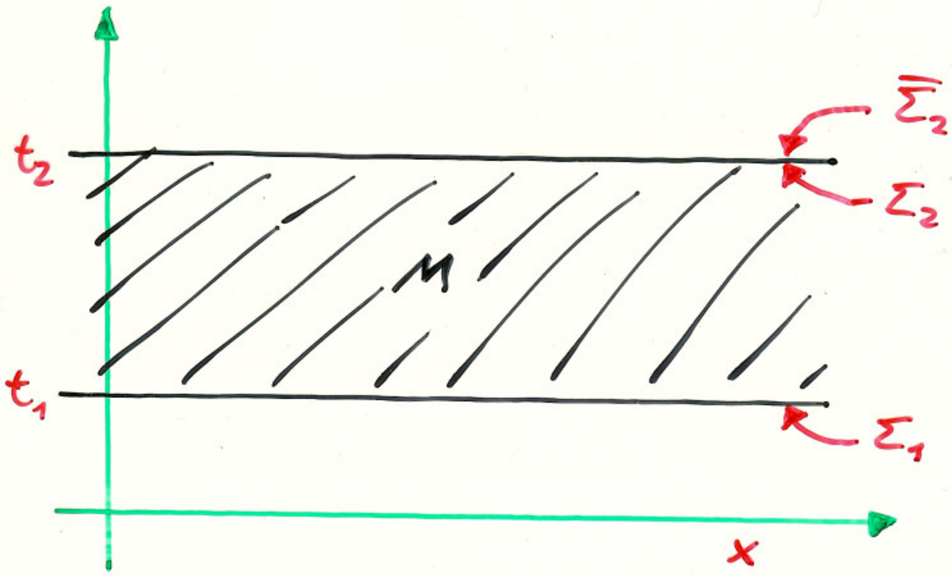
(T5)



$\tilde{\rho}_{M_1} : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\bar{\Sigma}}$ $\quad \tilde{\rho}_{M_2} : \mathcal{H}_{\bar{\Sigma}} \to \mathcal{H}_{\bar{\Sigma}_2}$ $\quad \tilde{\rho}_{M_1 \cup M_2} : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\bar{\Sigma}_2}$

composition property: $\tilde{\rho}_{M_1 \cup M_2} = \tilde{\rho}_{M_2} \circ \tilde{\rho}_{M_1}$

RECOVERING STANDARD QM



$M = [t_1, t_2] \times \mathbb{R}^3$

$\partial M = \Sigma_1 \cup \bar{\Sigma}_2$

$\mathcal{H}_{\partial M} = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\bar{\Sigma}_2}$

- due to time-translation symmetry $\mathcal{H}_{\Sigma_1} \simeq \mathcal{H}_{\Sigma_2}$
  → the standard state space of QM
- amplitude map $\rho_M : \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\bar{\Sigma}_2} \to \mathbb{C}$
  induces $\tilde{\rho}_M : \mathcal{H}_{\Sigma_1} \to \mathcal{H}_{\Sigma_2}$
  this is ordinary (finite) time-evolution operator
  $\rho_M(\psi \otimes \eta) = \langle \eta | \tilde{\rho}_M | \psi \rangle$
- (T4b) ensures unitarity
- (T5) ensures consistency of time composition
  $\tilde{\rho}_{[t_2, t_3]} \circ \tilde{\rho}_{[t_1, t_2]} = \tilde{\rho}_{[t_1, t_3]}$

GENERAL BOUNDARIES AND QUANTIZATION

How to produce quantum field theories of the general boundary form?

- CANONICAL approaches pose serious problems:
  - need to go beyond spacelike hypersurfaces/foliations
  - one-parameter description of (time) evolution no longer adequate
    → possible generalization: "local" Hamiltonian in the spirit of Tomonaga-Schwinger
    → see recent work of C. Rovelli, F. Conrady, L. Doplicher

- PATH INTEGRAL approaches seem much more amenable to the extension to general boundaries (unsurprisingly, see origins of TQFT):
  - employ path integrals on bounded regions together with Schrödinger representation, i.e. wave functions on boundaries

  → see hep-th/0505267, hep-th/0509122:
  successful implementation of certain regions with timelike boundaries in Klein-Gordon theory, including vacuum and all particle states

SCHRÖDINGER-FEYNMAN APPROACH

- use Schrödinger representation:
  states are wave functions on configuration space

  Σ hypersurface, K_Σ space of field configurations on Σ
  $\rightarrow$ $\mathcal{H}_\Sigma = C(K_\Sigma)$ space of functions on K_Σ

- inner product for $\psi, \psi' \in \mathcal{H}_\Sigma$

  $$\langle \psi, \psi' \rangle_\Sigma := \int_{K_\Sigma} \mathcal{D}\varphi \, \overline{\psi(\varphi)} \psi'(\varphi)$$

  integral over all field. conf. on Σ

- if hypersurface decomposes, $\Sigma = \Sigma_1 \cup \Sigma_2$
  $\rightarrow$ $\mathcal{H}_\Sigma = C(K_{\Sigma_1} \times K_{\Sigma_2}) = C(K_{\Sigma_1}) \otimes C(K_{\Sigma_2}) = \mathcal{H}_{\Sigma_1} \otimes \mathcal{H}_{\Sigma_2}$

- space of wave functions on hypersurface with opposite orientation is the same $\mathcal{H}_\Sigma \simeq \mathcal{H}_{\bar\Sigma}$,
  but wave functions for corresponding physical states are related by complex conjugation

  $$(\iota_\Sigma \psi)(\varphi) = \overline{\psi(\varphi)} \qquad \forall \varphi \in K_\Sigma \quad \psi \in \mathcal{H}_\Sigma$$

- Feynman path integral gives amplitude for spacetime region M with boundary Σ

  $$\rho_M(\psi) := \int_{K_\Sigma} \mathcal{D}\varphi \, \psi(\varphi) \, Z_M(\varphi)$$

  integral over boundary field conf.

  $$Z_M(\varphi) := \int_{\substack{K_M \\ \phi|_\Sigma = \varphi}} \mathcal{D}\phi \, e^{i S_M(\phi)}$$

  field propagator

  path integral over field configurations in the interior of M that match the boundary data φ

  classical action in spacetime region M

THE VACUUM

assume context where single vacuum state is sensible (generalizations are possible)

- for each oriented hypersurface Σ there is a unique vacuum wave function $\psi_{\Sigma,0}$

- compatibility with conjugation: the vacuum wave function on the oppositely oriented hyperplane is the complex conjugate

$$\psi_{\bar{\Sigma},0}(\varphi) = \overline{\psi_{\Sigma,0}(\varphi)} \qquad \forall \varphi \in K_\Sigma$$

- if hypersurface decomposes $\Sigma = \Sigma_1 \cup \Sigma_2$, vacuum wave function factorizes

$$\psi_{\Sigma,0}(\varphi_1, \varphi_2) = \psi_{\Sigma_1,0}(\varphi_1)\,\psi_{\Sigma_2,0}(\varphi_2)$$

- the vacuum wave function is normalized

$$\int_{K_\Sigma} \mathcal{D}\varphi \, |\psi_{\Sigma,0}(\varphi)|^2 = 1$$

- the vacuum has unit amplitude

$$\rho_M(\psi_{\Sigma,0}) = \int_{K_\Sigma} \mathcal{D}\varphi \, \psi_{\Sigma,0}(\varphi)\, Z_M(\varphi) = 1$$

region M with boundary Σ

► these properties imply:
vacuum is preserved under (time) evolution

KLEIN-GORDON QFT: TIME INTERVALS

consider real massive Klein-Gordon theory

$$S_M(\phi) = \frac{1}{2}\int_M dt\, d^3x \left(\partial_0\phi\, \partial_0\phi - \sum_i \partial_i\phi\, \partial_i\phi - m^2\phi^2\right)$$

standard case: region given by time interval $M = [t_1, t_2] \times \mathbb{R}^3$

- boundary decomposes into equal-time hyperplanes Σ_1, Σ_2 at t_1, t_2
- field configurations on Σ_t are field configurations in space

▸ propagator is

normalization factor

$$Z_M(\varphi_1, \varphi_2) = N e^{-\frac{1}{2}\int d^3x\, (\varphi_1\ \varphi_2)\, W \begin{pmatrix}\varphi_1\\ \varphi_2\end{pmatrix}}$$

config. at t_1 config. at t_2

$$W = \frac{-i\omega}{\sin\omega\Delta}\begin{pmatrix}\cos\omega\Delta & -1\\ -1 & \cos\omega\Delta\end{pmatrix}$$

$$\Delta = t_2 - t_1 \qquad \omega = \sqrt{-\sum_i \partial_i^2 + m^2}$$

▸ the vacuum can be obtained by demanding invariance under time-evolution and ansatz

$$\psi_0(\varphi) = C e^{-\frac{1}{2}\int d^3x\, \varphi A \varphi}$$

unknown operator

normalization factor

→ $A = \pm\omega$, conventional choice $A = \omega$

▸ 1-particle state using $\check{\varphi}(p) = 2E\int d^3x\, e^{ipx}\varphi(x)$

$$\psi_p(\varphi) = \check{\varphi}(p)\,\psi_0(\varphi)$$

amplitude $\rho(\psi_p \otimes \psi_{p'}) = (2\pi)^3\, 2E\, \delta^3(p-p')\, e^{-iE\Delta}$

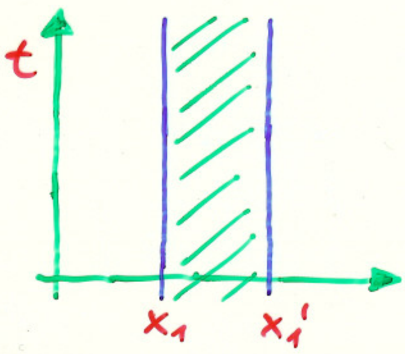
at t_1 at t_2

▸ 2-particle state

$$\psi_{p,p'}(\varphi) = \left(\check{\varphi}(p)\check{\varphi}(p') - (2\pi)^3\, 2E\, \delta^3(p+p')\right)\psi_0(\varphi)$$

KLEIN-GORDON QFT: GEN. HYPERPLANES

- obtain general <u>spacelike</u> hyperplanes through Lorentz boosts → induced vacuum, propagator, particle states
- more interesting are <u>timelike</u> hyperplanes



consider $M = \mathbb{R} \times [x_1, x_1'] \times \mathbb{R}^2$

boundary $\Sigma = \Sigma_1 \cup \Sigma_2$

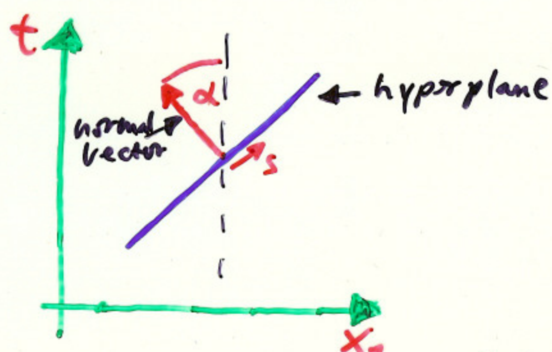
! • configuration space must be restricted to <u>physical</u> configurations, i.e. those that extend to classical solutions

• obtain vacuum through invariance under <u>spatial</u> evolution

$$\psi_0(\varphi) = C e^{-\frac{1}{2}\int dt\, d^2x\, \varphi K_1 \varphi}$$

with $K_1 := \sqrt{-\partial_0^2 + \sum_{i\geq 2} \partial_i^2 - m^2}$ this is well defined on physical configurations

- obtain vacuum on general hyperplanes through Lorentz boosts from considered special cases



• hyperplane coordinatized by s, x_2, x_3

• α is angle between normal vector and time axis

$$\psi_0(\varphi) = C e^{-\frac{1}{2}\int ds\, dx_2\, dx_3\, \varphi \tau \varphi}$$

with $\tau := \sqrt{-\partial_s^2 + \cos 2\alpha \left(-\sum_{i\geq 2} \partial_i^2 + m^2\right)}$

• null limit of spacelike and timelike vacuum is well defined and agrees

• vacuum wave function changes smoothly under <u>Euclidean</u> rotations in spacetime

- particle states (timelike case):

Fourier transform is

$$\check{\varphi}^{\pm}(E,\tilde{p}) := 2p_1 \int dt\, d^2\tilde{x}\; e^{\pm i(Et-\tilde{p}\tilde{x})}\, \varphi(t,\tilde{x})$$

energy with separated sign

2-momentum (p_2,p_3)

sign of energy distinguishes incoming versus outgoing particles

- 1-particle state

$$\psi^{\pm}_{E,\tilde{p}}(\varphi) = \check{\varphi}^{\pm}(E,\tilde{p})\,\psi_0(\varphi)$$

- 1-1 amplitude

$$\rho_{[x_1,x_1']}\left(\psi^{a}_{E,\tilde{p}} \otimes \psi^{a'}_{E',\tilde{p}'}\right) = (2\pi)^3\, 2p_1\, \delta_{a,-a'}\, \delta(E-E')\, \delta^2(\tilde{p}-\tilde{p}')\, e^{ip_1\Delta}$$

$\Delta = |x_1' - x_1|$ $\qquad p_1 := \sqrt{E^2 - \tilde{p}^2 - m^2}$

this means that an in-particle on one hyperplane can only pair with an out-particle on the other

- in multi-particle states individual particles may be independently incoming or outgoing, causality does not restict to one or the other

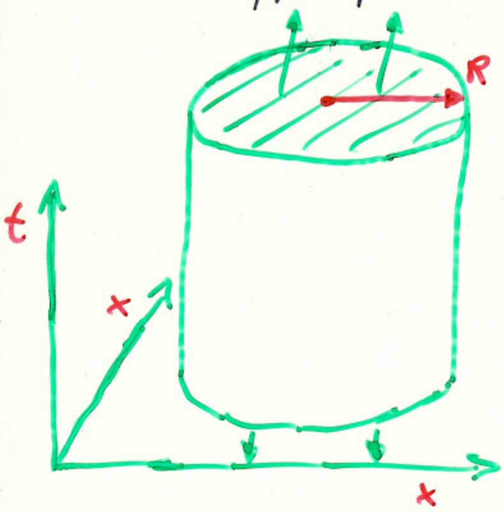
- 2-particle state

$$\psi^{a,a'}_{(E,\tilde{p}),(E',\tilde{p}')}(\varphi) = \Big(\check{\varphi}^{a}(E,\tilde{p})\,\check{\varphi}^{a'}(E',\tilde{p}') - (2\pi)^3\, 2p_1\, \delta_{a,-a'}\, \delta(E-E')\, \delta^2(\tilde{p}-\tilde{p}')\Big)\,\psi_0(\varphi)$$

KLEIN-GORDON QFT: HYPERCYLINDERS

hypercylinder: sphere in space time all of time $\Sigma = \mathbb{R} \times S^2$

full hypercylinder



nested hypercylinders

- configuration spaces must be restricted to physical ones

→ obtain consistent propagators

- obtain vacuum by "radial evolution" combined with asymptotic approximation of hyperplane case

"inside" vacuum wave function is

$$\psi^{I}_{R,0}(\varphi) = C\, e^{-\frac{1}{2}\int dt\, \underbrace{d\Omega\, 4\pi}_{\text{solid angle integral}}\, \varphi(t,\Omega)\, \underbrace{(B^I_R\, \varphi)}_{\text{operator}}(t,\Omega)}$$

find:

$$B^I_R = \frac{1 + i z^2 \left(j_\ell(z)\, j_\ell'(z) + n_\ell(z)\, n_\ell'(z)\right)}{p\, \left(j_\ell^2(z) + n_\ell^2(z)\right)}$$

$z := pR$

spherical Bessel functions

operator is defined in terms of its eigenvalues on mode expansion in terms of p and ℓ

total momentum

spherical harmonic mode

"outside" vacuum is then simply

$$\psi^{O}_{R,0}(\varphi) = \overline{\psi^{I}_{R,0}(\varphi)}$$

▶ particle states:

mode expansion of field

$$\check\varphi^{\pm}_{R,l,m}(E) = \int dt\, d\Omega\, 4\pi \frac{\sqrt{2}}{\sqrt{p}\,|h_l(pR)|} e^{\pm iEt} Y_l^{\mp m}(\Omega)\, \varphi(t,\Omega)$$

sign of energy encoded separately ($E>0$)

spherical harmonic of 3rd kind

spherical harmonics

- 1-particle state:

"outside" $\psi^{O,\pm}_{R,E,l,m}(\varphi) = \check\varphi^{\pm}_{R,l,m}(E)\, \psi^{O}_{R,0}(\varphi)$

"inside" $\psi^{I,\pm}_{R,E,l,m}(\varphi) = \check\varphi^{\mp}_{R,l,m}(E)\, \psi^{I}_{R,0}(\varphi)$

- 2-particle state:

$$\psi^{O,a,a'}_{R,(E,l,m),(E',l',m')}(\varphi) = \Big(\check\varphi^{a}_{R,l,m}(E)\, \check\varphi^{a'}_{R,l',m'}(E') - 8\pi^2 \delta(E-E')\, \delta_{l,l'}\, \delta_{m,m'}\, \delta_{a,-a'} \Big)\, \psi^{O}_{R,0}(\varphi)$$

- two different types of amplitudes:

– "transition" between hypercylinders, e.g. 1-1

$$\rho_{[R,\hat R]}\left(\psi^{I,a}_{R,E,l,m}, \psi^{O,a'}_{\hat R,E,l',m'}\right) = \alpha_{[R,\hat R],p,l}\, 8\pi^2 \delta(E-E')\, \delta_{l,l'}\, \delta_{m,m'}\, \delta_{a,a'}$$

with $\alpha_{[R,\hat R],p,l} := \dfrac{h_l(p\hat R)}{h_l(pR)} \dfrac{|h_l(pR)|}{|h_l(p\hat R)|}$

– solid hypercylinder "indecomposable" amplitude

$$\rho_R\left(\psi^{O,a,a'}_{R,(E,l,m),(E',l',m')}\right) = \frac{h_l(pR)}{\overline{h_l(pR)}}\, 8\pi^2 \delta(E-E')\, \delta_{l,l'}\, \delta_{m,m'}\, \delta_{a,-a'}$$

only in-out and out-in gives non-zero contribution