Causal Parametrices and Lorentzian spectral geometries without a classical time

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Motivation

Ultimate Aim: Does the dynamical coupling of quantum fields to geometry lead to a noncommutativity of spacetime ?
Strategy: Generalizing The notion of generally covariant quantum theory develop (and study) a notion of “local” quantum physics over (possibly) noncommutative spacetimes.

First step: Find a suitable notion of noncommutative spacetimes, which should be such that

- The Dirac-, resp. Klein-Gordon equation admit a well posed initial value problem.
- There should be a well-defined concept of isometric (...) embeddings (and hence also diffeomorphisms) of such spacetimes into each other.
- There exist appropriate notions of “causality” and “microlocal analysis”.

Plan of the talk

- 1 NCG-preliminaries I and II
- 2 LOST (“**LO**rentzian **S**pectral **T**riples”)
- 3 stably causal LOST
- 4 Ghyst (“**G**lobally **HY**perbolic **S**pectral **T**riples”)
- 5 outlook

NCG-Preliminaries I

- By the Gelfand-Naimark Theorem non-compact spaces correspond to nonunital (pre-)C^*-algebras, $\mathscr{L}$, e.g.

$$\mathscr{L} = C_0^\infty(M),\ C_c^\infty(M),\ \mathcal{S}(M),\ L_\infty^p(M),\ \ldots$$

- To describe Vector-bundles, like spinors, differential forms etc. one needs a "preferred unitalization" $\tilde{\mathscr{L}}$ of $\mathscr{L}$ (corresponding to some compactifaction of M.), e.g. $\tilde{\mathscr{L}} = \mathcal{B}^\infty(M)$.
  Vector-bundles are then given as $\mathscr{L}$-pullbacks of finitely generated projective $\tilde{\mathscr{L}}$-modules, i.e. they are given by $\mathscr{L}$-modules of the form $\mathscr{L} \cdot \mathcal{E}$, where

$$\mathcal{E} = \tilde{\mathscr{L}}^N p, \qquad p \in M_N(\tilde{\mathscr{L}}), \qquad p^2 = p = p^*.$$

- Note that elements $a \in \mathscr{L}$ can be used to localize objects on M (in particular if $\mathscr{L} = C_c^\infty$).
  E.g. if D is a differential operator of order 1, then aD will be a differential operator of order 1 with support in $supp(a)$ (with a discrete spectrum).

NCG-Preliminaries II

Consider the Dirac-Operator D on some d-dim. Lorentzian Spin-manifold (M, g, σ).
Schematically it is in every point of M $D = i\gamma^\mu \partial_\mu + \chi$ where $\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2g^{\mu\nu}$.
Note that $(\gamma^i)^* = \gamma^i$ but $(\gamma^0)^* = -\gamma^0 =: -\beta$.

- $\Rightarrow D^* = \beta D \beta$ globally!
- The operator $\langle D \rangle^2 := \frac{1}{2}(DD^* + D^*D)$ is locally of the form

$$\langle D \rangle^2 = -\tilde{g}^{\mu\nu} \partial_\mu \partial_\nu + \text{lower order terms}$$

  where $\tilde{g}^{\mu\nu}$ is a *Riemmanian metric* on M. Hence $\langle D \rangle^2$ is elliptic and in particular every eigenvalue has only a finite degeneracy.
- In fact

$$Tr_\omega(a\langle D\rangle^{-d}) := \lim_{N\to\infty} \frac{1}{\log N} \sum_{n=1}^{N} \mu_n(a\langle D\rangle^{-d}) \propto \int a d\nu_g \qquad \forall a \in L^1(M, g)$$

  independent of the choice of β.
  (Here $\mu_n(A)$ denote the characteristic values of the operator A.)

LOST

A Lorentzian spectral triple of dimension $d = n + 1$ is an ordered collection

$$\boldsymbol{L} = (\mathscr{L} \subset \tilde{\mathscr{L}}, \mathcal{H}, D, \beta, J, \gamma)$$

of 7 objects, *in particular* required to fullfill the following conditions:

- $\mathcal{H}$ is a Hilbert space,
  $\mathscr{L} \subset B(\mathcal{H})$ is a non-unital pre-C^*-algebra,
  $\tilde{\mathscr{L}} \subset B(\mathcal{H})$ a unitalization of $\mathscr{L}$.
  $[\beta, \mathscr{L}] = 0$ and $\beta^* = -\beta$ and $\beta^2 = -1$.
  $D : \mathrm{dom}(D) \to \mathcal{H}$ is a β-selfadj. op.,
  $[D, a]$, $[\langle D \rangle, a]$ and $[\langle D \rangle, [D, a]]$ are bounded $\forall\, a \in \mathscr{R}$.
- The space

  $$\mathcal{H}^\infty = \bigcap_k \mathrm{dom}(D^k)$$

  is the $\mathscr{L}$-pullback of a finitely generated projective $\tilde{\mathscr{L}}$-module.
- For all $a \in \mathscr{L}$ the operators $a\langle D \rangle^{-d}$ are compact and $0 < \mathrm{Tr}_\omega(a\langle D \rangle^{-n}) < \infty$.

still LOST

- J is an anti-unitary operator on $\mathcal{H}$ and $\gamma \in \mathscr{L}'$, fulfilling the relations:

$$J\mathscr{L}J \subset \mathscr{L}',$$

  + (anti-)commutation relations between γ, J, and β
- $[[D,a], JbJ] = 0 \qquad \forall\, a, b \in \mathscr{L}$
- $D\gamma = (-1)^n \gamma D$, and
- ......

Thm *Each Lorentzian Spin-manifold determines (not uniquely) a LOST.*

Conversely, if $\boldsymbol{L}$ is a LOST with $\mathscr{L} = C_c^\infty(M)$ for some finite-dimensional C^∞-manifold M, and $\tilde{\mathscr{L}} = \mathcal{B}^\infty(M)$, then $\boldsymbol{L}$ determines uniquely a Lorentzian Spin-manifold $\boldsymbol{M} = (M, g, \sigma)$.

Remark: The metric is reconstructed as

$$g(\mathrm{d}a, \mathrm{d}b) = \frac{1}{2}\{[D,a],[D,b]\}.$$

LOST for Moyal

Let $\Theta^{\mu\nu}$ be a skew-symmetric, nondegenerate 4×4-matrix. We define the algebra $\mathcal{S}_\Theta(\mathbb{R}^4)$ as the vector space $\mathcal{S}(\mathbb{R}^4)$ of smooth functions of rapid decay on $\mathbb{R}^4$, equipped with the $*_\Theta$-product

$$(f *_\Theta g)(x) = \frac{1}{\pi^4 \mathrm{det}\Theta} \int\int \ f(x+s)g(x+t) \exp\{-2is\Theta^{-1}t\} \ \mathrm{d}^4t\mathrm{d}^4s$$

Note that the $*_\Theta$-product can be extended to $\mathcal{B}^\infty(\mathbb{R}^4)$ (Gayral et al.).

- $\mathcal{S}_\Theta(\mathbb{R}^4)$ can now , via the $*_\Theta$-product, be represented on $L^2(\mathbb{R}^4, \mathbb{C}^4)$.
- Using the Tomita-Takesaki Theorem one can then construct an appropriate operator J.
- Together with the usual (i.e. "commutative") operators γ, β and $D = i\gamma^\mu\partial_\mu$, these data can be shown to fullfill the axioms for a LOST.

We shall call this example the Moyal-deformed Minkowski spacetime.

Stably causal LOST

Recall that a Lorentzian manifold (M,g) is called stably causal if there exists a function f on M, such that in every point of M one has $g(\mathrm{d}f,\mathrm{d}f)<0$.
f can then be used to define a global time coordinate t.

Definition: *A LOST $\boldsymbol{L}$ is called stably causal if*

- *There exists a selfadjoint operator ∂_0 on $\mathcal{H}$ such that*

$$[\partial_0, a] \in \tilde{\mathscr{L}} \qquad \forall a \in \tilde{\mathscr{L}} \qquad\qquad \left(\Rightarrow \mathcal{H}^\infty \subseteq \bigcap_k \mathrm{dom} \partial_0^k \right).$$

- *There exists a unitary element $u \in \tilde{\mathscr{L}}$ such that*

$$[\partial_0, u] = iu = -\beta[D,u]$$

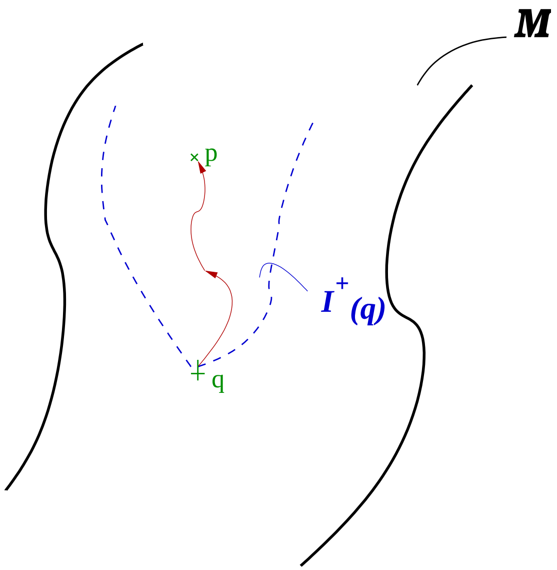
- $u^\alpha \in \tilde{\mathscr{L}}$ *for all* $\alpha \in \mathbb{R}$ *($\Rightarrow$ spec(∂_0)$=\mathbb{R}$).*

Remark: Thus, in the classical case, $g(\mathrm{d}x^0,\mathrm{d}x^0) = [D,x^0]^2 = \beta^2 = -1$ with $u =: e^{ix^0}$.

more stably causal LOST

Remark:

The Moyal-deformed Minkowski spacetime is stably causal. For $\Theta^{0i} \neq 0$ one has $[t, \mathcal{S}_q^2] \neq 0$.



Obviously ∂_0 generates a one-paramter family of automorphisms of $\tilde{\mathscr{L}}$, i.e. a "time flow". Setting $U_t := e^{i\partial_0 t}$ they are $\alpha_t(a) = U_t a U_{-t}$.

Of course, if they exist at all, there are (in general) infinitely many pairs (∂_0, u) admissable for the data of a given LOST $\boldsymbol{L}$, corresponding to the different foliations. Set

$$\mathcal{T}_{\boldsymbol{L}} := \{U_t = e^{i\partial_0 t} \mid \quad (\partial_0, u) \quad \text{admissable for} \quad \boldsymbol{L}\,\}$$

Remark: Given one pair (∂_0, u), individual members of $\mathcal{T}_{\boldsymbol{L}}$ are of the form $U_t = V e^{i\partial_0 t} V^*$. Where the unitaries V on $\mathcal{H}$ fullfill in particular $V \tilde{\mathscr{L}} V^* \subset \tilde{\mathscr{L}}$ and $V D V^* = D$.

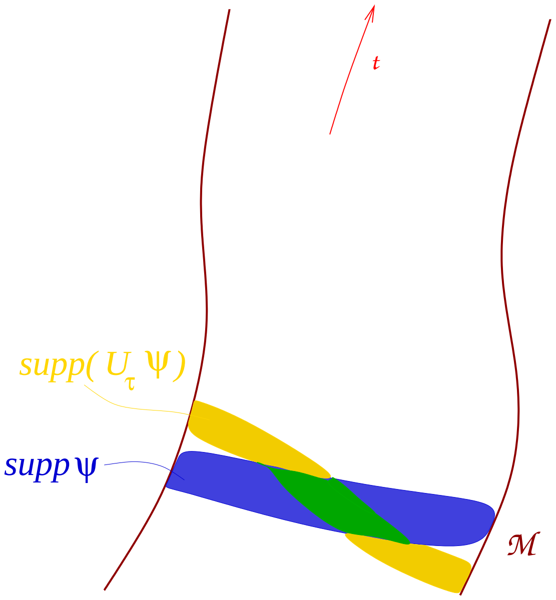
Lost – compactness

Definition: Let

$$V := \{\psi \in \mathcal{H}^\infty \mid \ \mathrm{supp_t}\langle\psi|U_t\psi\rangle \text{ is compact } \forall U_t \in \mathcal{T}_{\boldsymbol{L}}\}.$$

$$\mathcal{H}_c^\infty := \{\psi \in V \mid \ \mathrm{supp_t}\langle\psi|U_t\varphi\rangle \text{ is compact } \forall U_t \in \mathcal{T}_{\boldsymbol{L}},\ \varphi \in V\}.$$

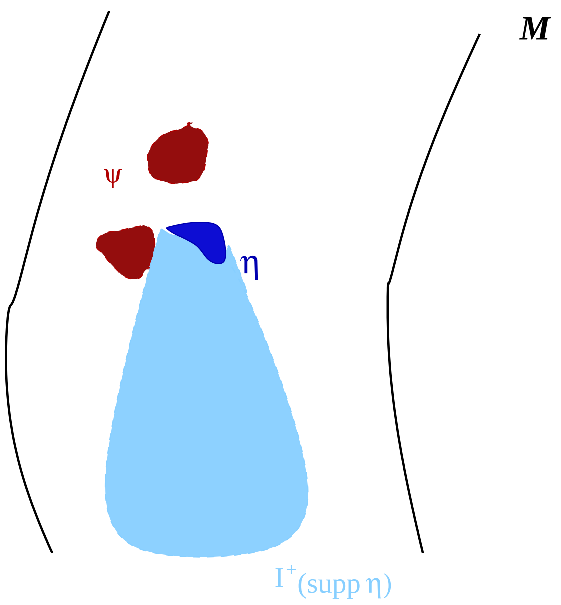
(In the commutative case, elements of $\mathcal{H}_0^\infty$ correspond to Spinors of compact support.)
$\Rightarrow \ \mathscr{L}\mathcal{H}_0^\infty \subset \mathcal{H}_0^\infty$ and $D\mathcal{H}_0^\infty \subset \mathcal{H}_0^\infty$



to precede (Ghysts)

Definition:

Let $(\mathcal{H}_0^\infty)'$ be the space of linear continuous functionals on $\mathcal{H}_0^\infty$. Given two elements $\xi, \eta \in (\mathcal{H}_0^\infty)'$, we say that ξ precedes η, $\eta \prec \xi$ if and only if, *for every* $\psi \in \mathcal{H}_0^\infty$ such that for all $U_t \in \mathcal{T}_{\boldsymbol{L}}$ one has (pictorially)



$$(*) \quad \Rightarrow \quad \mathrm{supp}\psi \subset I^-(\mathrm{supp}\eta)^c$$

$$(**) \quad \Rightarrow \quad \mathrm{supp}\xi \subset J^-(\mathrm{supp}\eta)$$

Analogously one defines a notion of succeeding, $\eta \succ \xi$.

to causally propagate (Ghysts)

Given a (stably causal) LOST $\boldsymbol{L} = (\mathscr{L} \subset \tilde{\mathscr{L}}, \mathcal{H}, D, \beta, J, \gamma, (\partial_0, u))$ we can now seek for maps

$$E : \mathcal{H}_0^\infty \to (\mathcal{H}_0^\infty)'$$

fullfilling the equation "$D(E\psi) = \psi$" for all testfunctions $\psi \in \mathcal{H}_0^\infty$, resp.

$$\langle E\psi, D\varphi \rangle = \langle \psi, \varphi \rangle \qquad \forall \psi, \varphi \in \mathcal{H}_0^\infty .$$

Maps $E^\pm$ of this type, which additionally obey

$$\psi \succ E^+\psi \qquad \text{resp.} \qquad \psi \prec E^-\psi \qquad \forall \psi \in \mathcal{H}_0^\infty$$

will be called the advanced (resp. retarded) Parametrices of D.

Remark: Such maps $E, E^\pm$ do not exist on every Lorentzian manifold, and if they exist, they are usually not unique. On globally hyperbolic spacetimes they exist uniquely.

Ghysts

Definition:

A globally hyperbolic spectral triple is a stably causal LOST $\boldsymbol{L}$ such that there exist uniquely determined advanced and retarded Parametrices $E^{\pm}$.

Remark:

- Thus every globally hyperbolic Spin-manifold gives rise to a Ghyst.
- The Moyal deformed Minkowski spacetime is a Ghyst
- “Spectral quadruples (with a commutative time) give rise to Ghysts.

Outlook

- One could try to find (necessary and) sufficient conditions for a stably causal LOST to be a Ghyst.
  E.g. if $H := D - \beta\partial_0$ is essentially selfadjoint, the existence and uniqueness of the causal parametrices can be (constructively) proven.
  However, even on classical globally hyperbolic spacetimes the Dirac-Hamiltonian is rarely essentially selfadjoint (for generic foliations).
- One should try to reformulate the axioms for Ghysts by only using the causal parametrices rather than the extremly unwieldy operators $\langle D \rangle$.
- $\rightarrow$ Generalize more tools/concepts from Microlocal Analysis to our setup. (E.g. “Bicharacteristics”)
- Infer the geodesic distance directly from $E^{\pm}$ (à la Hadamard).
- Find more examples (Isospectral deformations, q-deformations, κ-deformaed Minkowski spacetime, The DFR-quantum spacetime...)
- enter QFT......