Oct 2005

2-dimensional open-closed TQFT from a state sum model

H Pfeiffer (AEI)
A Lauda (DPMMS Cambridge)

Introduction

TQFT

Manifolds with corners

closed TQFT: global vs local picture

open-closed TQFT: global vs local picture

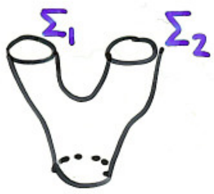
TQFT (Topological Quantum Field Theory)

$$Z(\Sigma_f) \quad \text{vector space}$$

$$\uparrow Z(M) \quad \text{linear map} \qquad \xrightarrow[\text{TQFT}]{Z}$$

$$Z(\Sigma_i) \quad \text{vector space}$$

$\partial M = \Sigma_i^* \,\dot\cup\, \Sigma_f$

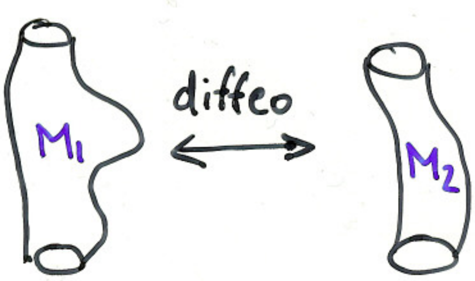


$$Z(\Sigma_1 \,\dot\cup\, \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2)$$

$Z($  $) = Z($  $) \otimes Z($  $)$

$M_1 \quad M_2 \qquad\qquad M_1 \qquad\qquad M_2$

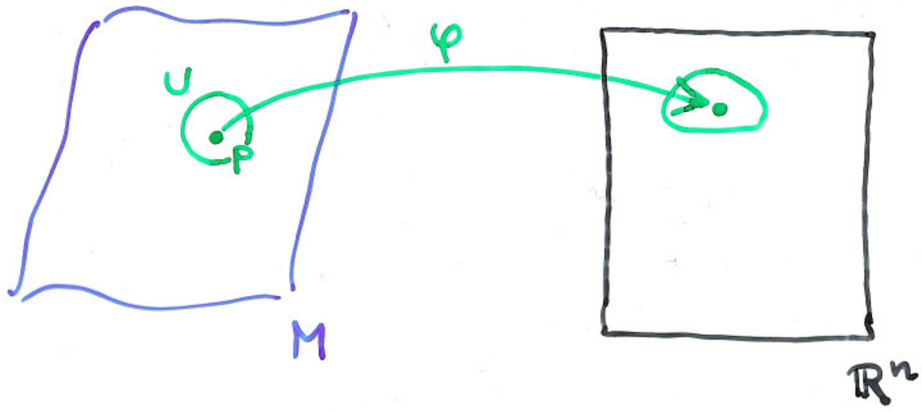
$Z($  $) = Z($  $) \circ Z($  $)$



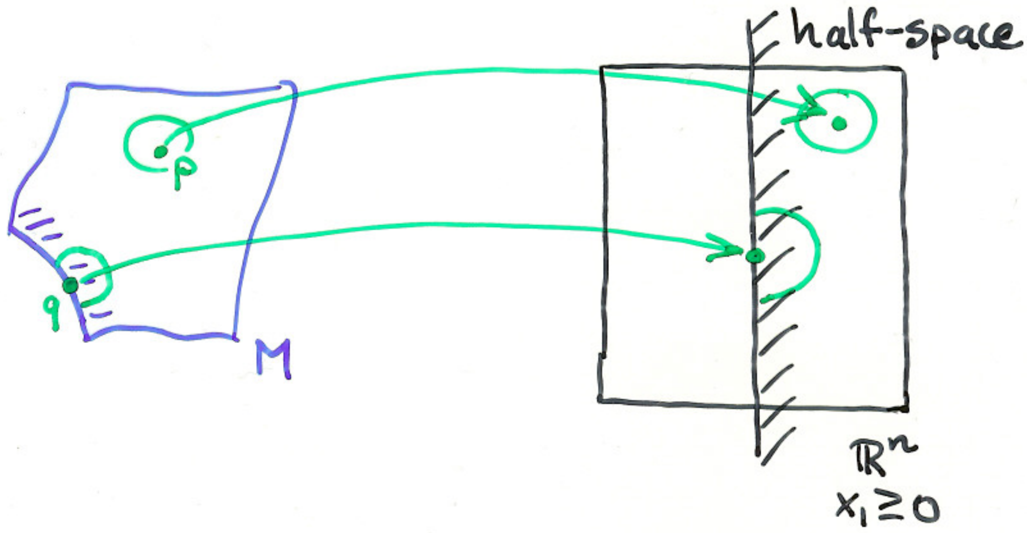
$$\Rightarrow \quad Z(M_1) = Z(M_2)$$

Manifolds with corners

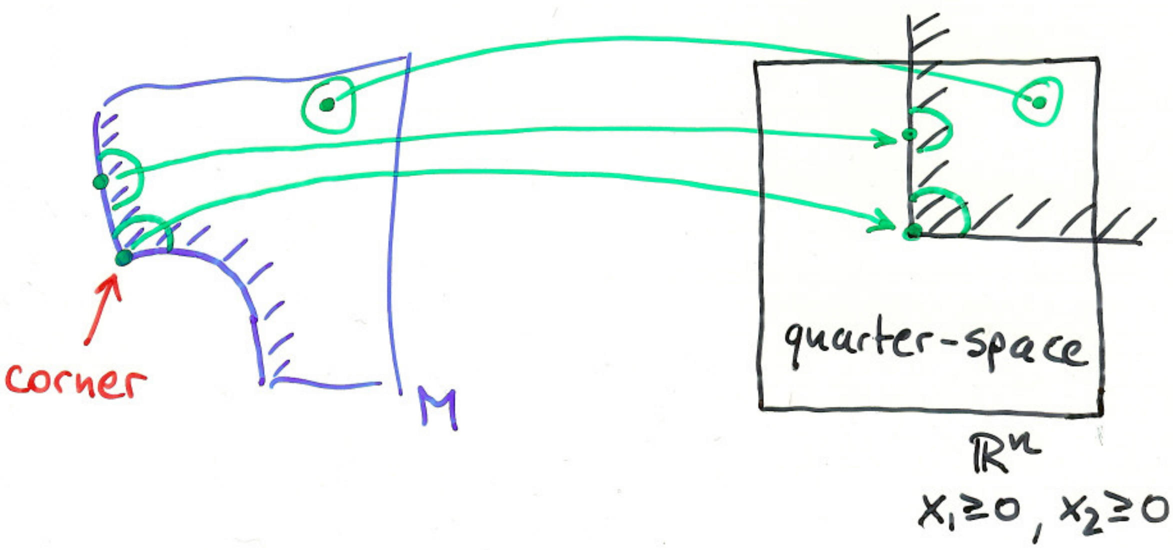
without boundary



with boundary

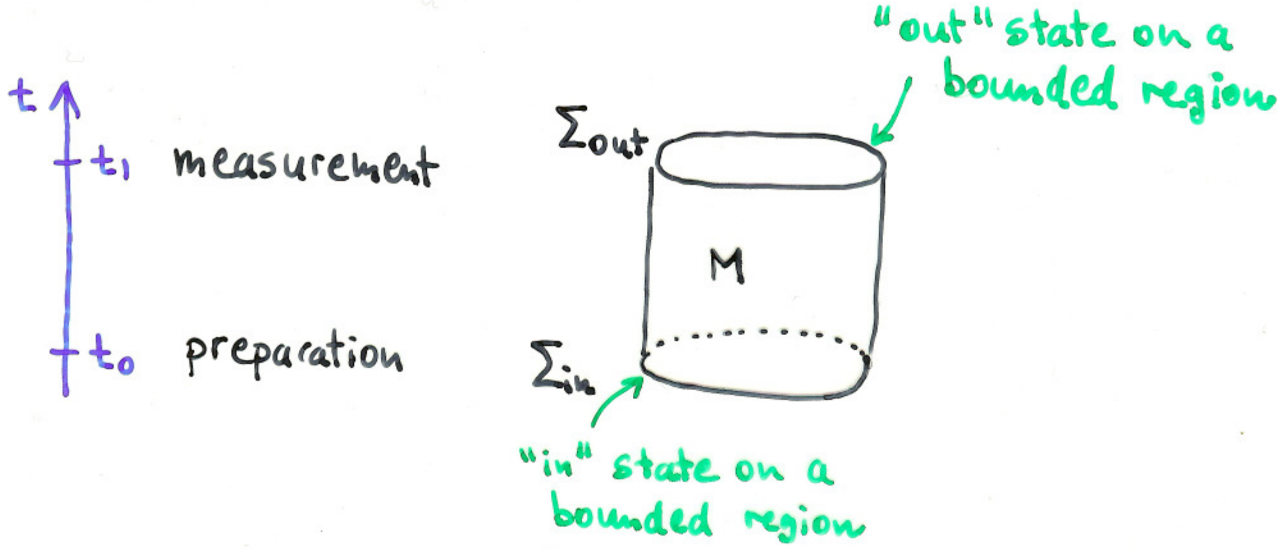


with corners



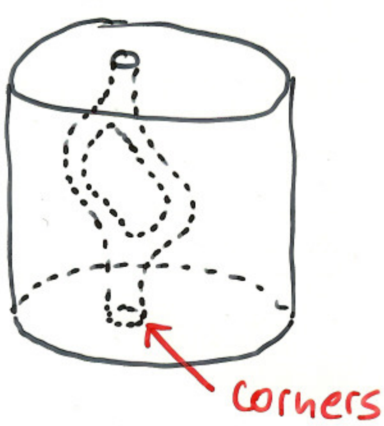
Why corners ?

1) realistic experiments



→ want a manifold M such that "∂M" is a manifold with boundary !

2) want to insert particle world lines with general quantum numbers → cut out a tube

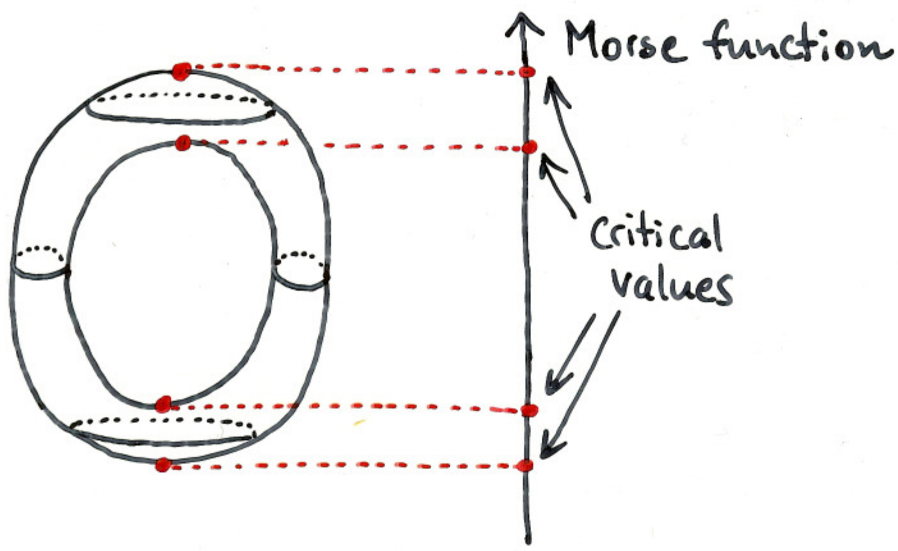


3) It makes the algebraic structures more transparent, see Conclusion !

2-dimensional TQFT

Closed case

$C := Z(S^1)$ k-vector space



building blocks (= generators):

$k \xrightarrow{\eta} C$

$C \otimes C \xrightarrow{\mu} C$

$C \xrightarrow{\varepsilon} k$

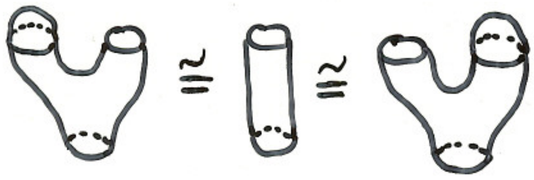
$C \xrightarrow{\Delta} C \otimes C$

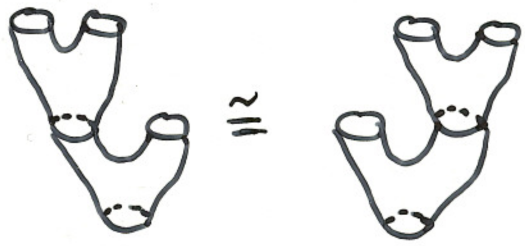
also need:

$C \xrightarrow{\mathrm{id}} C$

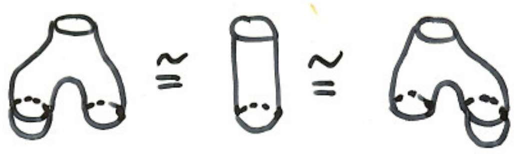
$C \otimes C \xrightarrow{\tau} C \otimes C$

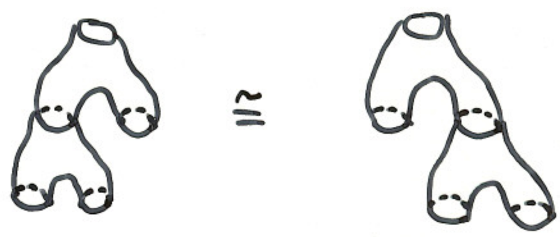
diffeomorphisms (= relations)



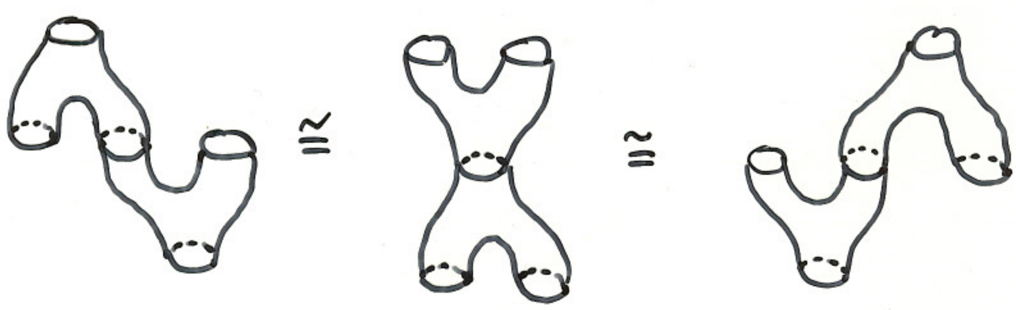


→ (C, μ, η) associative algebra with unit

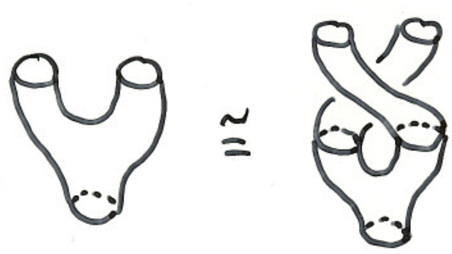




→ (C, Δ, ε) coassociative coalgebra with counit



→ Frobenius algebra



→ commutative

[Abrams '96, Sawin '95]: global structure is a commutative Frobenius algebra

Can you construct 2d TQFTs from state sums?

Thm: [Fukuma–Hosono–Kawai '94]
Yes. Sometimes.

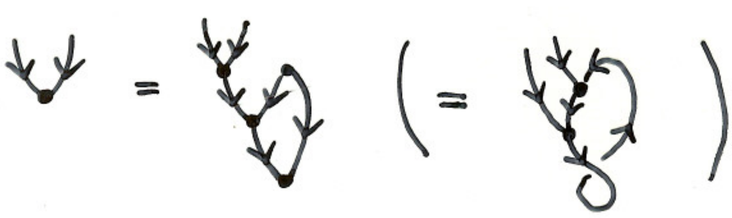
A semisimple associative algebra

The symmetric invariant bilinear form

$g(a,b) :=$  $:= tr_A(L_a \circ L_b)$ $\quad L_a(b) := \mu(a \otimes b)$

is non-degenerate.

$\Rightarrow$  exists such that 

$\Rightarrow$ 

Frobenius algebra structure:

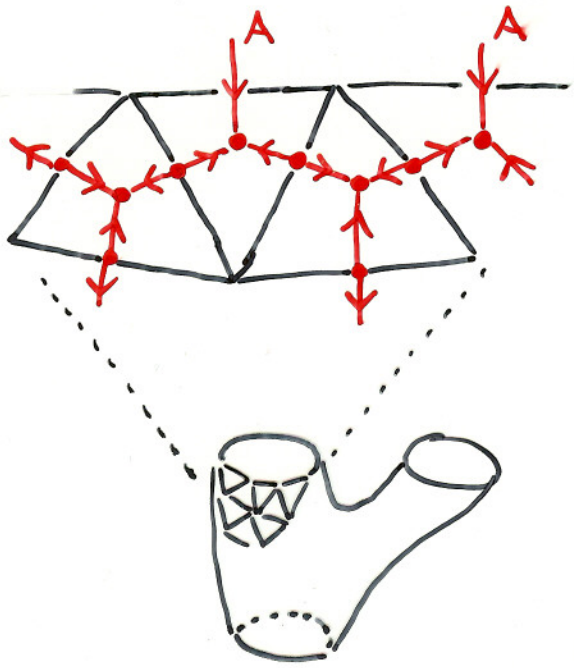
facts:

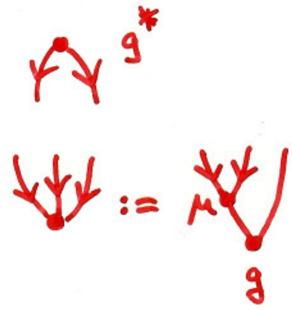
1) A is symmetric  but not in general commutative 

2) bubble move 

Fukuma-Hosono-Kawai state sum:

Triangulate the 2-manifold with boundary, compute

$Z :=$ 



Thm: Z is invariant under Pachner moves.

(bubble move) (associativity)

2d TQFT has two descriptions: (semisimple case)

global: Commutative Frobenius algebra $Z(S^1)$

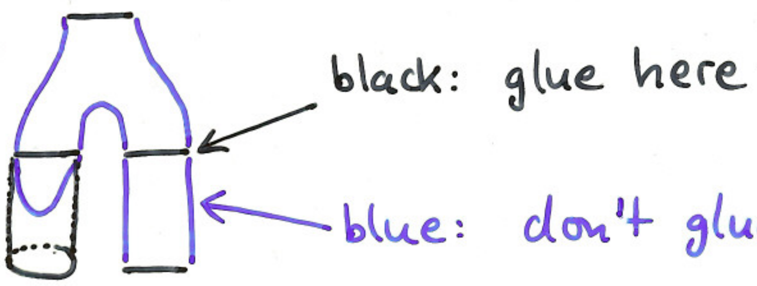
local: symmetric Frobenius algebra A

$\longrightarrow \; Z(S^1) = Z(A)$

partition fn centre

How do I know this ??

Study 2-manifolds with corners with the following additional structure:



<u>open-closed case</u>

S^1 (closed string) $\qquad C := Z(S^1)$

$I = [0,1]$ (open string) $\qquad A := Z(I)$

$\Big\}$ k-vector spaces

<u>generators</u>

$C \to C \otimes C$

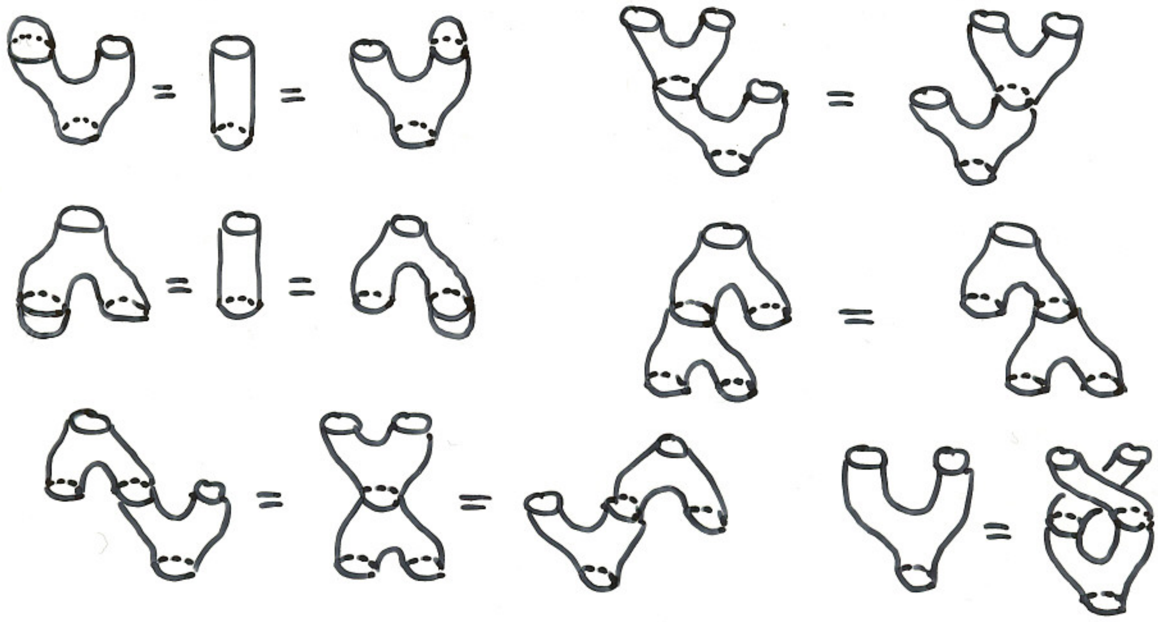
$A \to A \otimes A$

$A \xrightarrow{\iota} C$

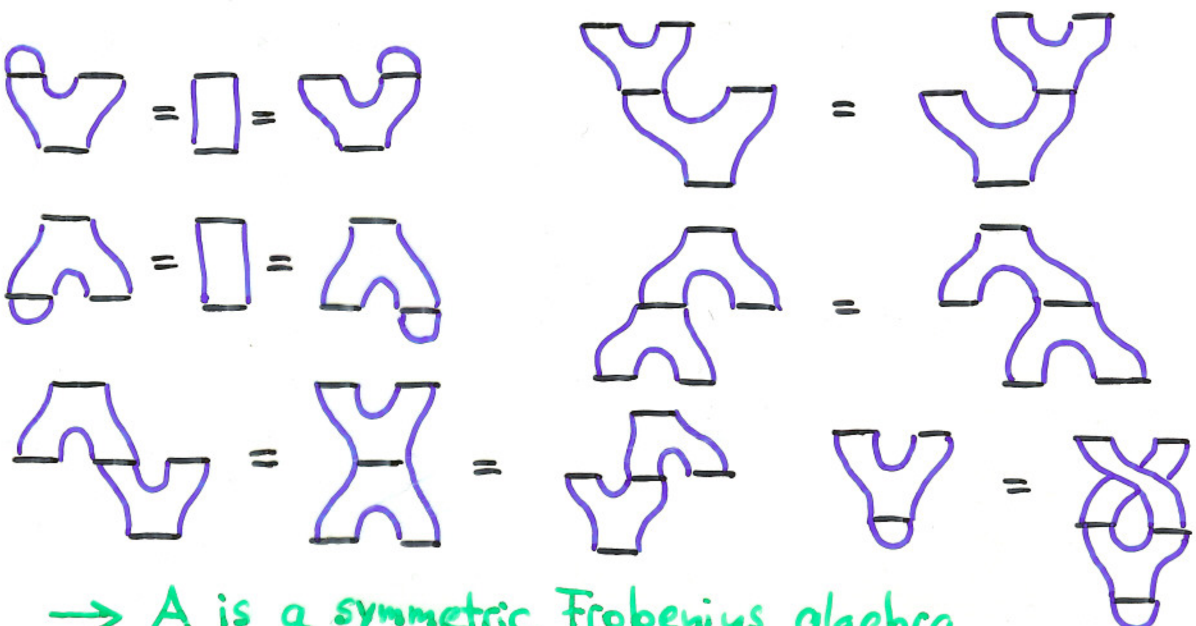
$C \xrightarrow{\iota^*} A$

<u>also</u>:

relations



→ C is a commutative Frobenius algebra

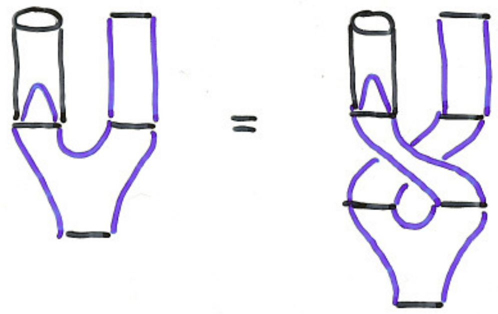


→ A is a symmetric Frobenius algebra

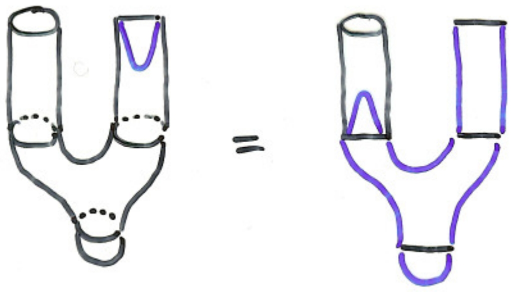
→ $\iota: C \to A$ is a homomorphism of algebras

(11)

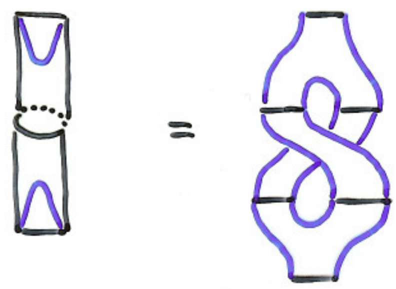
relations, continued:



→ $\iota: C \to A$ maps into the centre of A



→ $\iota^*: A \to C$ is dual to $\iota: C \to A$



"Cardy condition"

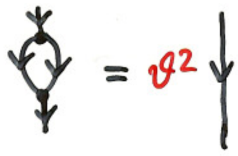
Thm: The global structure is described by (C, A, ι, ι^*) + conditions

[Cardy, Lewellen, Moore-Segal, AL-HP]

State sum with corners [AL-HP]

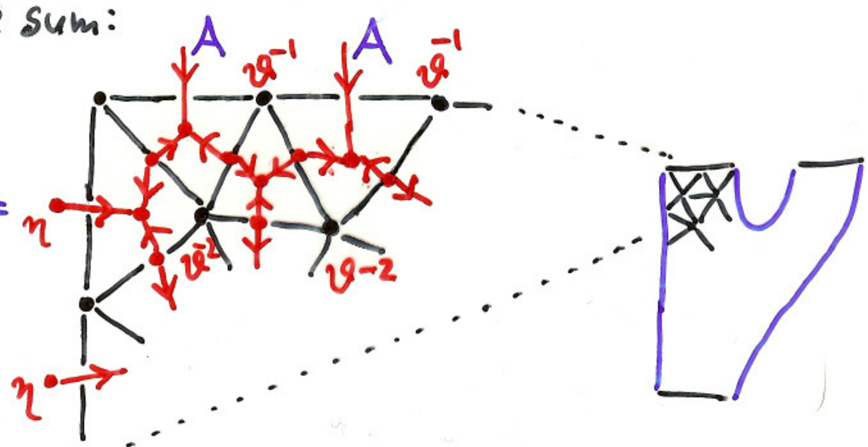
A semisimple associative algebra

$:= \frac{1}{\vartheta^2}\, tr_A (L_a \circ L_b)$ rescale the bilinear form by $1/\vartheta^2$, $\vartheta \neq 0$.

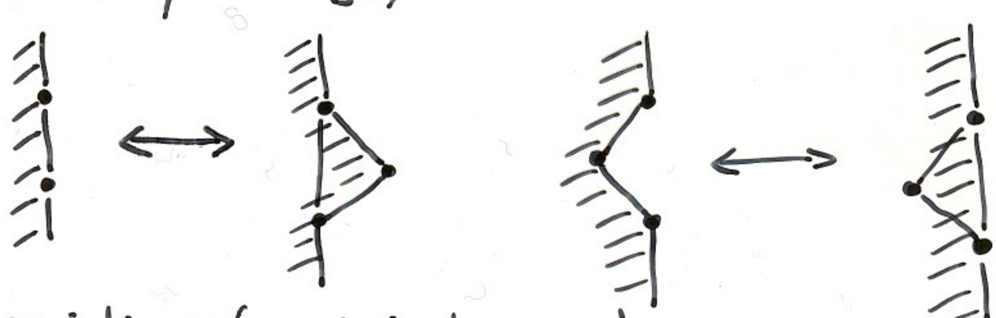
$\Rightarrow$ 

weak bubble move

state sum:

$Z :=$ 

Thm: Z is invariant under Pachner moves in the interior (bistellar moves) and under boundary Pachner moves (elementary shellings)



Two descriptions (semisimple case)

global: $Z(I)$, $Z(S^1)$, ι, ι^* + conditions

$$\begin{array}{ll} \text{global:} & Z(I) \quad Z(S^1) \\ & \;\; \| \qquad\;\; \| \\ \text{local:} & \;\, A \qquad Z(A) \end{array}$$

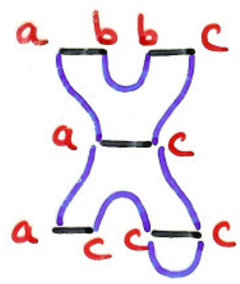
agree !

Conclusion:

- If you study only the closed sector, you have secretly "closed" all open strings 
  and taken the centre $A \to Z(A)$.

  This has obscured the structure ...

- Include the corners in order to make the full structure (A, C, ι, ι^*) visible !

$\to$ can include boundary labels 

get A_{ab} rather than just A

and recover $A_{ab} := \mathrm{Hom}(a, b)$ of a Vect_K-enriched category.

"open strings with D-branes"