Asymptotically Safe Quantum Gravity :

Emergence of Fractal Spacetimes

M. Reuter

Classical General Relativity:

successful phenomenological theory at laboratory, solar system, galactic, ... (?) length scales $\ell \gg \ell_{Pl} \equiv \sqrt{G} \approx 10^{-33}$ cm

Quantized General Relativity:

perturbatively nonrenormalizable $\rightsquigarrow$

widespread belief that Quantum Einstein Gravity is merely an <u>effective</u> theory, applicable for $\ell \gg \ell_{Pl}$, rather than a <u>fundamental</u> (microscopic) theory valid at arbitrarily small distances.

- Usually a theory is considered "fundamental" if it is perturbatively renormalizable, i.e. if its infinities can be absorbed by redefining only <u>finitely</u> many parameters $(m, g, \cdots)$.

- Perturbatively <u>non</u>renormalizable theories:

  increasing number of counter terms as the loop order increases:

  $\rightsquigarrow$ infinitely many free parameters

  $\rightsquigarrow$ no predictive power

However :

There exist fundamental theories which are not perturbatively renormalizable :

- "nonperturbatively renormalizable" along the lines of K. Wilson's general principles of renormalization
- constructed by performing the infinite-cutoff limit at a non-Gaussian RG fixed point ($u_* \neq 0$)

  (pert. theory: trivial (Gaussian) fixed pt. $u_* = 0$.)

"fundamental" := infinite-cutoff limit (continuum limit) exists

Weinberg's "asymptotic safety" conjecture (1979):

Perhaps Quantum Einstein Gravity can be defined nonperturbatively at a non-Gaussian fixed point.

$d = 2+\varepsilon$: FP known to exist

$d = 4$: progress hampered by lack of appropriate calculational scheme

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→ Use "effective average action" which seems ideally suited. Wetterich 1993

Effective average action for gravity: M.R. 1996

$$\Gamma_k [g_{\mu\nu}, \cdots]$$

The Effective Average Action Γ_k

- Wilson-type (coarse grained) free energy functional

- IR cutoff at k: Γ_k contains the effect of all quantum fluctuations with momenta $p > k$, not (yet) of those with $p < k$.

- modes with $p < k$ suppressed in the path integral by $(\text{mass})^2 = R_k(p^2)$

- $\Gamma_{k \to \infty} = S$, classical (bare) action
  $\Gamma_{k \to 0} = \Gamma$, standard effective action

- Γ_k satisfies exact RG equation; symbolically:

$$\text{"}\; k\,\partial_k \Gamma_k = \frac{1}{2}\,\mathrm{Tr}\left[\left(\delta^2\Gamma_k + R_k\right)^{-1} k\,\partial_k R_k\right] \;\text{"}$$

- Powerful nonperturbative approximation scheme:
  "truncate" the space of action functionals,
  project RG flow onto finite dimensional subspace

Construction of Γ_k for Gravity

M.R. 1996

- starting point: $\int \mathcal{D}\gamma_{\mu\nu}\, e^{-S[\gamma_{\mu\nu}]}$
- decompose $\gamma_{\mu\nu} = \underbrace{\bar{g}_{\mu\nu}}_{\text{fixed backgrd. metric}} + h_{\mu\nu}$
- add background gauge fixing $S_{gf}[h;\bar{g}]$ + ghost terms
- expand $h_{\mu\nu}$ in $\bar{D}^2$-eigenmodes, and introduce IR cutoff k^2: only modes with generalized momenta ($\bar{D}^2$-eigenvalues) are integrated out.
- add sources: generating fctl. $W_k[\text{sources}; \bar{g}]$

$$W_k[\text{sources};\bar{g}] \xrightarrow{\text{Legendre transf.}} \Gamma_k[g_{\mu\nu}, \bar{g}_{\mu\nu}, \text{ghosts}]$$

$g_{\mu\nu} \equiv \langle \gamma_{\mu\nu} \rangle$

- derive exact RG equation from path integral:

$$k\frac{\partial}{\partial k}\Gamma_k[g,\bar{g},\cdots] = \mathrm{Tr}(\cdots)$$

- "Ordinary" diffeomorphism invariant action:

$$\Gamma_k[g] = \Gamma_k[g, \bar{g}=g, \text{ghosts}=0]$$

Arena of RG dynamics: Theory Space

theory space = space of "all" action functionals of a given symmetry type

$\cup$

$$A[g_{\mu\nu}, \cdots] = \sum_n \bar{g}_n \, I_n[g_{\mu\nu}, \cdots]$$

$\bar{g}_n$: generalized couplings

$\{I_n\}$: "basis" of diffeom. inv. functionals

$\ni \int\sqrt{g}, \ \int\sqrt{g}\,R, \ \int\sqrt{g}\,R^n, \cdots$, nonlocal terms

Theory space is coordinatized by essential, dimensionless couplings

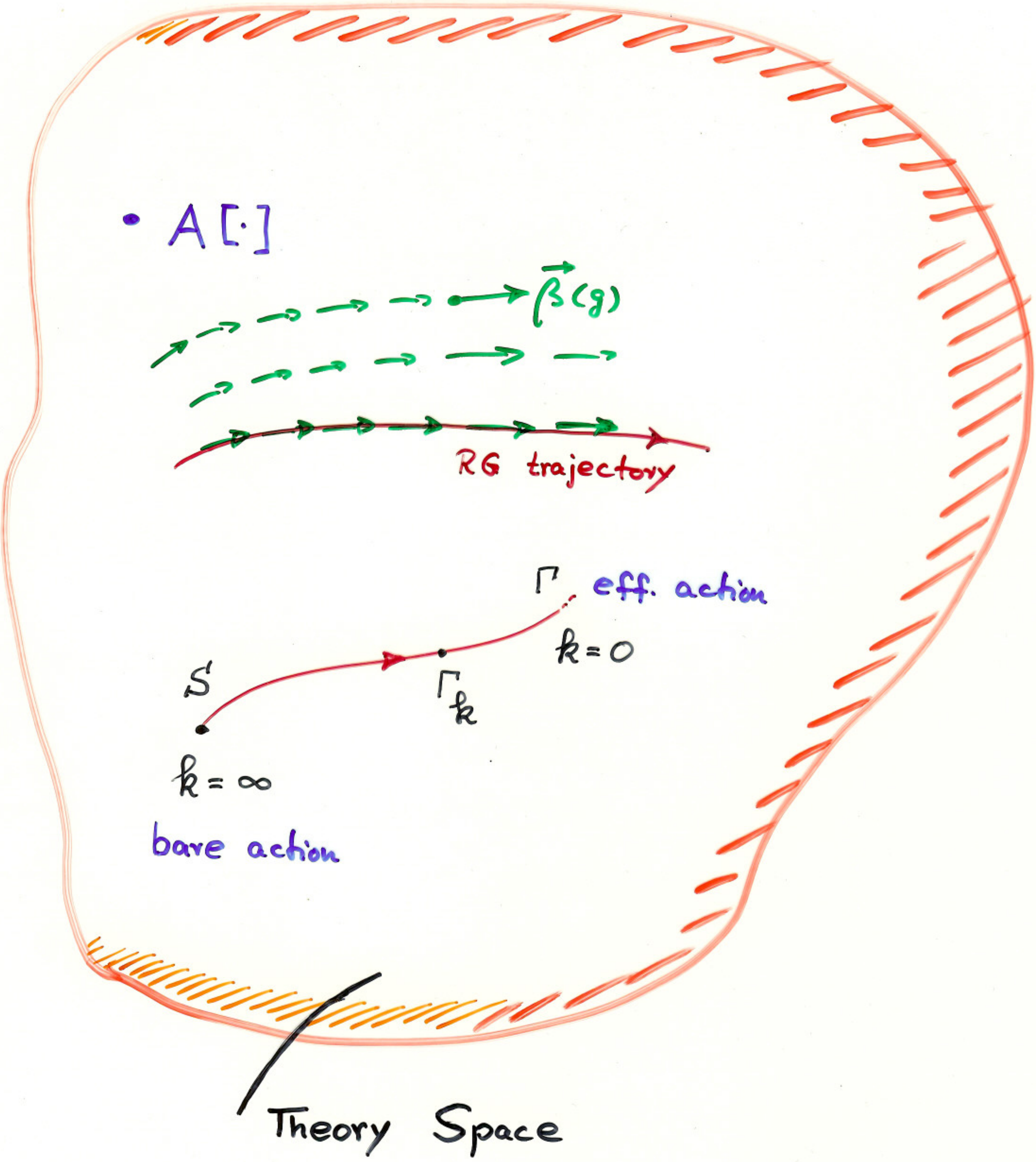
$$\boxed{g_n = \bar{g}_n \,/\, k^{\dim \bar{g}_n}}$$

Expansion $\Gamma_k[g, \cdots] = \sum_n \bar{g}_n(k) \, I_n[g, \cdots]$

yields RG eq. in component form:

$$\boxed{k\frac{d}{dk} g_n(k) = \beta_n(g_1, g_2, \cdots)}$$

$\vec{\beta} = (\beta_n)$: vector field on theory space ("RG flow")



$A[\cdot]$
$\vec{\beta}(g)$
RG trajectory
Γ eff. action
$k=0$
Γ_k
S
$k=\infty$
bare action
Theory Space

Nonperturbative construction of QEG

If there exists a non-Gaussian Fixed Point Γ_*,
$\vec{\beta}(\Gamma_*) = 0$, Quantum Einstein Gravity is nonperturbatively renormalizable ("asymptotically safe").

Weinberg 1979

Quantum theory is defined by a RG trajectory running inside the UV-critical hypersurface of the FP, with

initial point $= \Gamma_{k\to\infty} \equiv S$ = action infinitesimally close to Γ_*

end point $= \Gamma_0 \equiv \Gamma$

Note: Γ_* has no reason to be of Einstein-Hilbert form:

No quantization of General Relativity!

The Einstein-Hilbert Truncation

ansatz:

$$\Gamma_k = -\frac{1}{16\pi G_k}\int d^dx \sqrt{g}\,\{R - 2\Lambda_k\}$$

+ classical gauge fixing and ghost terms

two running parameters:

Newton constant G_k , dimensionless: $g(k) = k^{d-2} G_k$

cosmological constant Λ_k , dimensionless: $\lambda(k) = \Lambda_k / k^2$

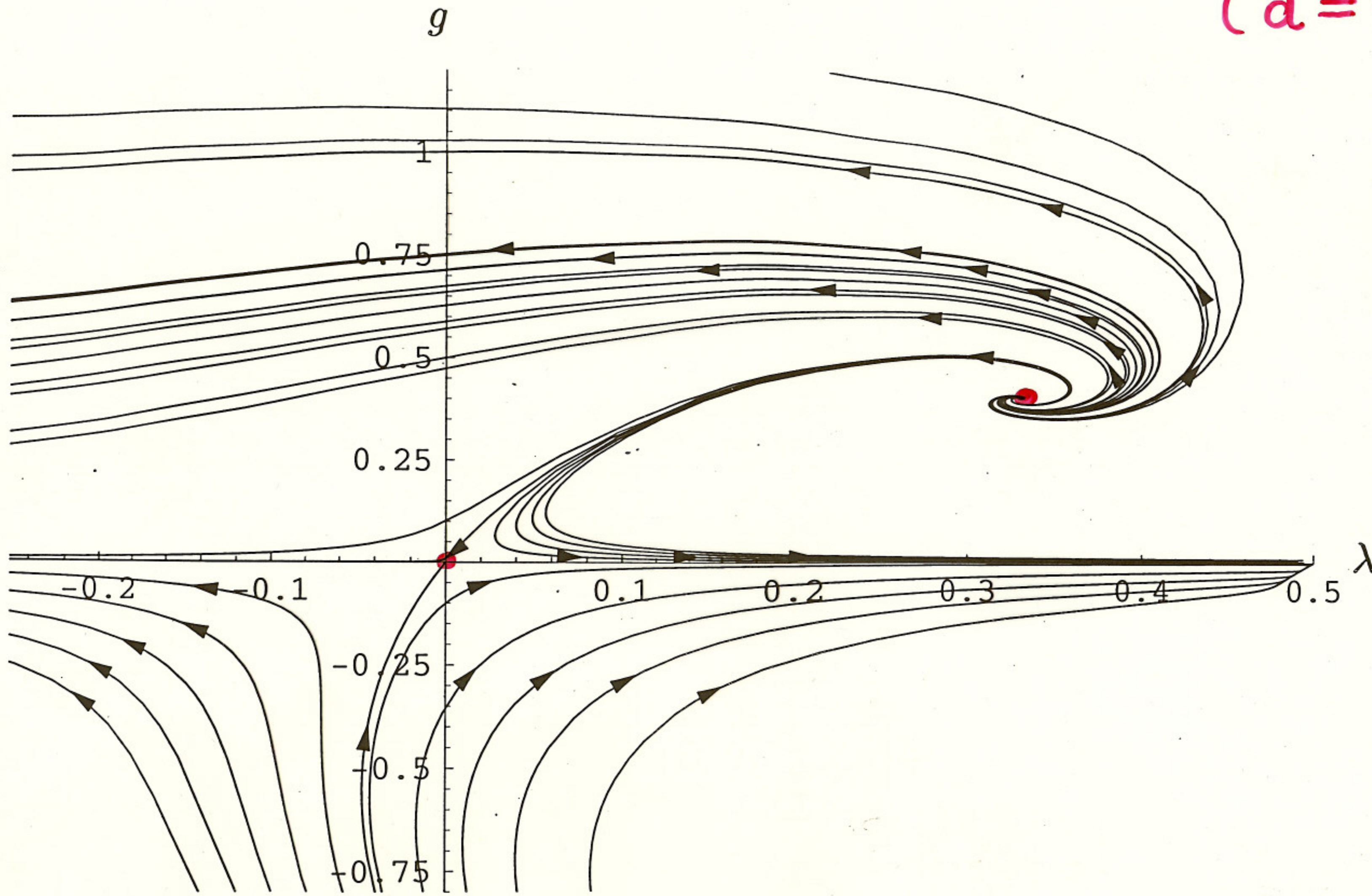
insert ansatz into flow equation, expand

$$\mathrm{Tr}[\cdots] = (\cdots)\int\sqrt{g} + (\cdots)\int\sqrt{g}R \quad + \text{unimportant}$$

$$\leadsto \quad \boxed{\begin{aligned} k\,\partial_k\, g(k) &= \beta_g(g,\lambda) \\ k\,\partial_k\, \lambda(k) &= \beta_\lambda(g,\lambda) \end{aligned}}$$

RG - Flow in the Einstein - Hilbert Truncation

(d=4)



M.R., F. Saueressig, hep-th/0110054

Checking the reliability of the Einstein-Hilbert truncation

(1) Cutoff scheme dependence ($\equiv \mathcal{R}_k(\cdot)$ - dependence) within the Einstein-Hilbert truncation

(critical exponents, $g_* \cdot \lambda_*$, ... scheme independent in the exact theory)

O. Lauscher, M.R., hep-th/0108040
M.R., F. Saueressig, hep-th/0110054
hep-th/0206145

PT-ERGE: A. Bonanno, M.R., hep-th/0410191

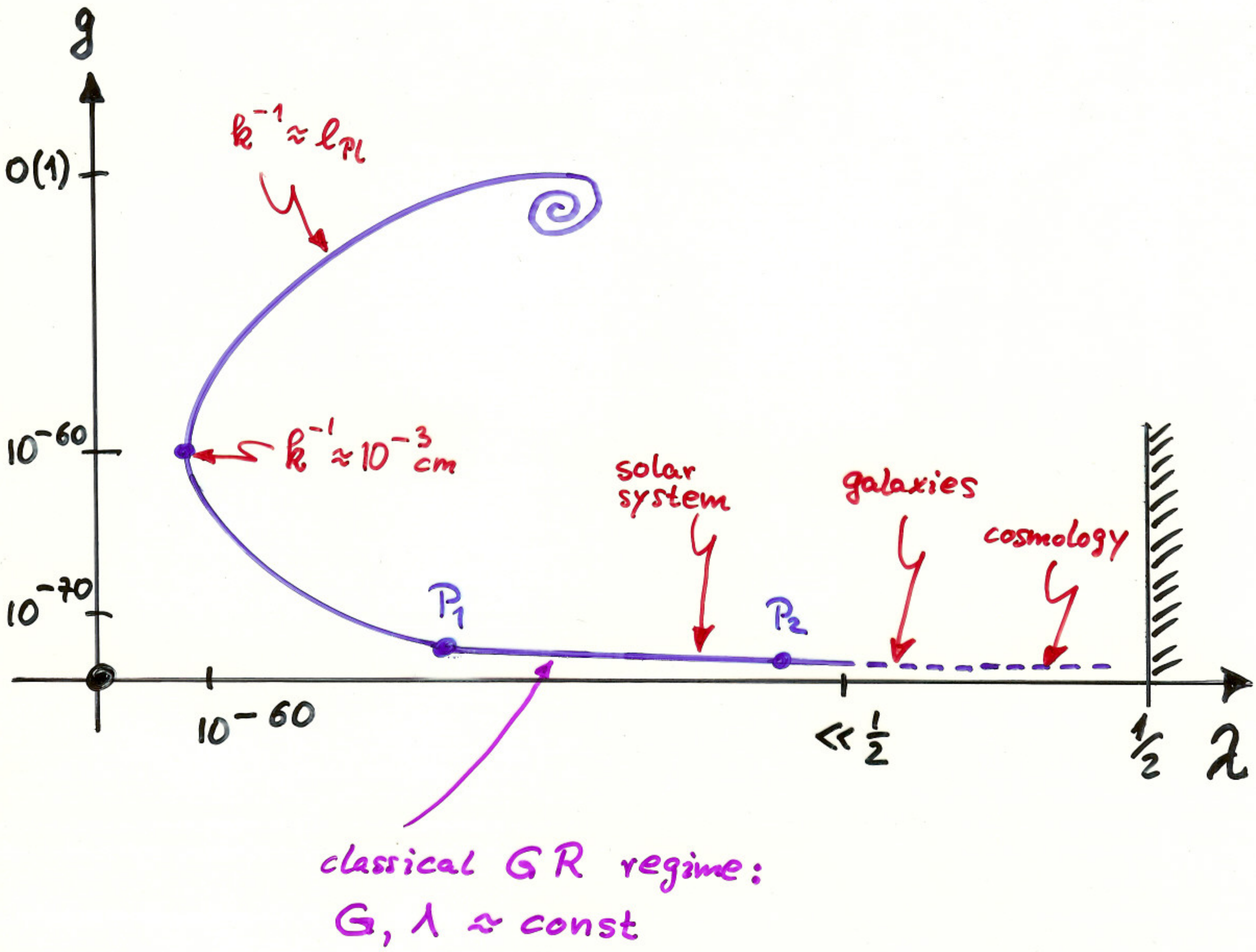
(2) Generalized truncation with R^2-invariant

O. Lauscher, M.R., hep-th/0110021
hep-th/0205062

$\Longrightarrow$ Vicinity of NGFP seems to be very well described by the EH-truncation.

$\hat{=}$ Nontrivial evidence supporting the hypothesis of a NGFP in the exact theory.

The RG trajectory realized in Nature



"Today" in cosmology:

$$\lambda_{cosmo} \equiv \frac{\Lambda_{cosmo}}{k^2_{cosmo}} = O(1) \quad !!!$$

$\Lambda_{cosmo} \approx H_0^2$

$k^2_{cosmo} \approx H_0^2$

Γ_k as an Effective Field Theory

→ Non-gauge theories in flat space:

- $\Gamma_k[\phi]$ generates correlation fcts. of fields averaged over spacetime volume of radius $\ell \approx k^{-1}$
- In particular: $\frac{\delta \Gamma_k}{\delta \phi(x)}[\langle\phi\rangle_k] = 0$

yields mean field $\langle\phi\rangle_k \equiv$ field "seen" by microscope with resolving power ℓ

- For observable $\mathcal{O}(\phi)$ involving only momenta $\approx k$:

$$\langle \mathcal{O}(\hat{\phi}) \rangle \approx \mathcal{O}(\langle\phi\rangle_k)$$

→ In gravity:

- averaging replaced by cutting-off of $\bar{D}^2$-eigenvalues
- relationship $\ell = \ell(k)$ more complicated in general

Metrics on QEG Spacetimes

- Fix a quantum theory by picking a specific RG trajectory $k \mapsto \Gamma_k[\cdot]$
- Solve eff. field eq. at any k:

$$\frac{\delta \Gamma_k}{\delta g_{\mu\nu}}\left[\langle g \rangle_k\right] = 0$$

scale dependent mean field

- A single trajectory gives rise to infinitely many "on-shell metrics":

$$\left\{ \langle g_{\mu\nu}(x) \rangle_k \;\middle|\; k = 0, \cdots, \infty \right\}$$

- Metric structure of "QEG spacetime" is described by <u>infinitely many</u> classical Riemannian metrics.

<u>Interpretation</u>: Observing spacetime under a microscope of resolving power ℓ_1 one sees a classical manifold with metric

$$\langle g_{\mu\nu} \rangle_{k_1} \quad \text{where} \quad \ell_1 = \ell(k_1)$$

- Metric is scale dependent!

<u>Analogy</u>: The length of the coast line of England depends on the size of the yardstick used to measure it.

How to find the relationship $\ell = \ell(k)$

recall: $\Gamma_k[g] = \Gamma_k[g\,;\ \bar{g} = g\ ;\ \text{ghosts} = 0]$

$\hookrightarrow \bar{D}^2 = D^2$

$-\bar{D}^2$ eigenvalues cut off at k^2

Algorithm:

- Given $\{\langle g_{\mu\nu}\rangle_k\}$, construct "on-shell Laplacians"

$$\Delta(k) \equiv D^2(\langle g\rangle_k)$$

- Find eigenfunction $\psi(x)$ of $-\Delta(k)$ with eigenvalue k^2

- Analyze $\psi(x)$:

  $\ell :=$ typical length scale displayed by $\psi(x)$ (e.g. period);
  proper length with respect to $\langle g_{\mu\nu}\rangle_k$ itself.

$\Longrightarrow \ell = \ell(k)$

QEG Spacetimes are Fractals

O. Lauscher,
M.R., 2002

... on scales at which the RG trajectory is near the FP 

$$\Lambda(k) = \lambda_* k^2 , \quad G(k) = g_* k^{2-d} , \ldots$$

Effective field eqs. in the Einstein-Hilbert approx.:

$$\boxed{R_{\mu\nu}(\langle g \rangle_k) = \Lambda(k) \langle g_{\mu\nu} \rangle_k}$$

- no dim.ful constants of integration
  $\rightsquigarrow$ solution has radius of curvature
  $$r_c(k) \propto \Lambda(k)^{-\frac{1}{2}} \propto k^{-1}$$

- $k \propto 1/\ell$ in the FP regime

$\Rightarrow$ Radius of curvature detected by "microscope of resolution ℓ":

$$\boxed{r_c(\ell) \propto \ell}$$

"Zooming" deeper into the spacetime structure (lowering ℓ) does not change the image seen by the microscope.

Dynamical Dimensional Reduction

O. Lauscher, M.R., 2002

$$\Gamma_k \supset \frac{1}{G(k)} \int d^d x \sqrt{g}\, R \qquad \rightsquigarrow$$

graviton propagator in d-dim. flat space $\propto \dfrac{G(k)}{p^2}$

Physical cutoff scale is $k = \sqrt{p^2}$ if $p^2 \in$ FP-regime.

$\Rightarrow$ dressed propagator (in $\Gamma \equiv \Gamma_0$):

$$\boxed{\mathcal{G}(p) \propto \frac{G(\sqrt{p^2})}{p^2}}$$

Anomalous dimension: $\quad \eta \equiv k \dfrac{d}{dk} \ln G(k)$

Example: FP regime

$$G(k) = g_* \, k^{2-d}$$

$$\Rightarrow \boxed{\eta = \eta_* \equiv 2-d} = \text{const}$$

$$d=4 : \quad \eta_* = -2$$

If η approximately constant

$$\Rightarrow \quad G(k) \propto k^{\eta}$$

$$\Rightarrow \quad \mathcal{G}(p) \propto \left(\frac{1}{p^2}\right)^{1-\eta/2}$$

$\eta \neq \eta_*$ $\qquad\qquad$ $\eta = \eta_*$

$$\mathcal{G}(x,y) \propto \frac{1}{|x-y|^{d+\eta-2}} \qquad\qquad \mathcal{G}(x,y) \propto \ln|x-y|$$

$\Rightarrow$ η has standard interpretation ($\rightarrow$ crit. phenomena)

$\Rightarrow$ effective dimensionality $= d + \eta(k)$

- At the FP: 2D-like logarithmic propagator

$$\boxed{d + \eta_* = d + (2-d) = 2 \quad \forall d}$$

- In the classical regime:

$$\boxed{d + \eta \approx d}$$

$\Rightarrow$ **Dynamical change of dimension $2 \rightarrow d$ during the RG evolution from the fixed point to the classical regime.**

The Spectral Dimension

Random walk (diffusion) of scalar test particle on classical, d-dimensional Riemannian manifold:

$$\partial_T K_g(x,x';T) = \Delta_g K_g(x,x';T)$$

$$\Delta_g \phi \equiv g^{-\frac{1}{2}} \partial_\mu \left(g^{1/2} g^{\mu\nu} \partial_\nu \phi \right)$$

Average return probability:

$$P_g(T) = \frac{1}{V} \int d^dx \sqrt{g(x)}\, K_g(x,x;T) = \frac{1}{V} \operatorname{Tr} e^{T \Delta_g}$$

$$\underset{\text{as. exp.}}{=} \left(\frac{1}{4\pi T} \right)^{d/2} \sum_{n=0}^{\infty} A_n T^n$$

Recover d from P_g: $\quad d = -2 \dfrac{d \ln P_g(T)}{d \ln T}$

Motivates the following definition for the spectral dimension of a QEG spacetime:

$$P(T) \equiv \langle P_\gamma(T) \rangle$$

$$\equiv \int \mathcal{D}\gamma_{\mu\nu} \mathcal{D}C \mathcal{D}\bar{C}\; P_\gamma(T)\; e^{-S[\gamma, C, \bar{C}]}$$

$$\boxed{\mathcal{D}_s \equiv -2 \frac{d \ln P(T)}{d \ln T}}$$

Evaluation of $\mathcal{D}_s$ in the RG framework

O. Lauscher
M.R. 2005

hep-th/0508202

(1) Solution of $R_{\mu\nu}(\langle g\rangle_k) = \Lambda(k)\langle g_{\mu\nu}\rangle_k$ satisfying

$$\langle g_{\mu\nu}\rangle_k = \frac{\Lambda(k_0)}{\Lambda(k)} \langle g_{\mu\nu}\rangle_{k_0}$$

leads to "running Laplacian" $\Delta(k) \equiv D^2(\langle g\rangle_k)$ satisfying

$$\boxed{\Delta(k) = \frac{\Lambda(k)}{\Lambda(k_0)} \Delta(k_0)}$$

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(2) • Γ_k = eff. field theory valid at k $\rightsquigarrow$

$$\langle \mathcal{O}(\gamma_{\mu\nu})\rangle \approx \mathcal{O}(\langle g_{\mu\nu}\rangle_k)$$

if $\mathcal{O}$ involves only momenta $\approx k$.

• Apply to $\mathcal{O} = \Delta(k)\, K(\cdots)$;

How to choose k ?

(3) If the diffusion process involves only momenta near a single, fixed k:

$$\partial_T K(x,x';T) = \Delta(k)\, K(x,x';T)$$

solved by

$$K(x,x';T) = \sum_n \phi_n(x)\, \phi_n(x')\, e^{-F(k^2)\,\mathcal{E}_n T}$$

where
$$-\Delta(k_0)\,\phi_n = \mathcal{E}_n\, \phi_n$$
$$F(k^2) \equiv \Lambda(k) / \Lambda(k_0)$$

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(4) Time evolution of initial probability distribution $p(x;0)$:

$$p(x;T) = \int d^d x' \sqrt{g_0(x')}\, K(x,x';T)\, p(x';0)$$

If $p(x;0) = \sum_n C_n\, \phi_n(x)$ then

$$p(x,T) = \sum_n C_n\, \phi_n(x)\, e^{-F(k^2)\,\mathcal{E}_n T}$$

If $C_n \neq 0$ only for a single $\mathcal{E}_N$:

single-scale problem, obvious choice is $k = \sqrt{\mathcal{E}_N}$

(5) If $C_n \neq 0$ for many different $\mathcal{E}_n$'s :

multi-scale problem,

choose mode-depent scale $\boxed{k = \sqrt{\mathcal{E}_n}}$ $\Rightarrow$

$$p(x,T) = \sum_n C_n \phi_n(x)\, e^{-\mathcal{F}(\mathcal{E}_n)\,\mathcal{E}_n T}$$

- <u>Example flat space</u>:

$$\phi_n \equiv \phi_p \sim e^{i p_\mu x^\mu}, \quad \mathcal{E}_n \equiv \mathcal{E}_p = p_\mu^2$$

Mode ϕ_p has "resolving power" $\ell \approx \frac{1}{|p|} = \frac{1}{\sqrt{\mathcal{E}_p}} \stackrel{!}{=} \frac{1}{k}$

and therefore "sees" metric $\langle g_{\mu\nu}\rangle_k$ with $k = \sqrt{\mathcal{E}_p}$.

- $k = \sqrt{\mathcal{E}_n}$ correct also in curved space since, by construction, k^2 cuts off the spectrum of D^2.

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(6) Traced propagation kernel (av. return probability):

$$P(T) = \frac{1}{V} \sum_n e^{-F(\mathcal{E}_n)\,\mathcal{E}_n T}$$

$$= \frac{1}{V} \mathrm{Tr}\, e^{F(-\Delta(k_0))\,\Delta(k_0)\,T}$$

Let k_0 = macroscopic scale, with $\Lambda(k_0) \ll m_{Pl}^2$,
and $\langle g_{\mu\nu} \rangle_{k_0} \approx$ flat metric :

$$\boxed{P(T) = \int \frac{d^d p}{(2\pi)^d}\, e^{-p^2 F(p^2)\, T}}$$

$$F(p^2) \equiv \Lambda(k = \sqrt{p^2}) / \Lambda(k_0)$$

$T \to \infty$: Long random walks, probe spacetime structure at large distances; governed by $F(p^2)$ with $p^2 \to 0$: <u>classical regime</u>

$T \to 0$: short random walks, probe spacetime structure at small distances; governed by $F(p^2)$ with $p^2 \to \infty$: <u>fixed point regime</u>

Spectral Dimension $\mathcal{D}_s = -2 \dfrac{d \ln P(T)}{d \ln T}$

- classical regime

no running, $\Lambda(k) \approx \Lambda(k_0)$, $\mathcal{F} \approx 1$

$$\Rightarrow \quad P(T) \propto T^{-d/2}$$

$$\Rightarrow \quad \boxed{\mathcal{D}_s = d}$$

- Fixed Point regime

$\Lambda(k) \propto k^2$, $\mathcal{F}(p^2) \propto p^2$

$$\Rightarrow \quad P(T) \propto T^{-d/4}$$

$$\Rightarrow \quad \boxed{\mathcal{D}_s = \tfrac{1}{2} d}$$

$d = 4$:

QEG, based upon any RG trajectory, predicts a continuous change of the fractal dimensionality of spacetime from $\mathcal{D}_s = 4$ at macroscopic to $\mathcal{D}_s = 2$ at microscopic distances.

Remarks:

- Result for $\mathcal{D}_s$ is actually an <u>exact</u> consequence of asymptotic safety and does not rely on any truncation.

- $d=4$ is special :

	$d+\eta$	$\mathcal{D}_s$
macroscopic	d	d
microscopic	2	$\frac{d}{2}$

agrees iff $d=4$!

- RG prediction of 2 dimensional fractal $(d=4)$ supported by Monte Carlo results :

$$\mathcal{D}_s(T\to\infty) = 4.02 \pm 0.1$$

$$\mathcal{D}_s(T\to 0) = 1.80 \pm 0.25$$

Ambjørn, Jurkiewicz, Loll, 2005

Summary

A QEG spacetime "manifold" carries infinitely many metrics $\langle g_{\mu\nu} \rangle_k$ which describe its metric structure on different coarse graining scales.

Spacetime has fractal properties in all regimes with nontrivial RG running of the gravitational couplings.

As a direct and exact consequence of asymptotic safety (existence of a FP) one finds a continuous change of its fractal dimension, from 4 at macroscopic distances to 2 on microscopic scales.