Effective Action from Geometric Formulation of Quantum Mechanics

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Geometric formulation of Quantum Mechanics

- First consider the space of functions $F|_\Psi = \langle \Psi, \hat{F}\Psi \rangle$, and think of the mean values $\langle \Psi, \hat{q}^n \Psi \rangle$ and $\langle \Psi, [\hat{q}^n, \hat{p}]_+ \Psi \rangle$ as the coordinates of this manifold.
- Equip the manifold with the metric G and symplectic structure Ω by use of the Hermitian inner product structure of the Hilbert space

$$\langle \Phi, \Psi \rangle = \frac{1}{2\hbar} G(\Phi, \Psi) + \frac{i}{2\hbar}\Omega(\Phi, \Psi), \quad G(\Phi, \Psi) = \Omega(\Phi, i\Psi)$$

- Define now $\Psi \sim \Phi$ the equivalence of two states when the mean values $\langle \Psi, \hat{q}\Psi \rangle = \langle \Phi, \hat{q}\Phi \rangle$ and $\langle \Psi, \hat{p}\Psi \rangle = \langle \Phi, \hat{p}\Phi \rangle$ coincide.
- The Hilbert space is characterized as a infinite dimensional Kähler manifold that is a Fiber Bundle: Classical phase space + vertical directions.

Coordinates, Vector Fields and the Poisson bracket

- Base manifold coordinates: $x^i = \langle \hat{q} \rangle, \langle \hat{p} \rangle$. Fiber coordinates:

$$G^{i_1 \dots i_n} = \langle (\hat{x}^{(i_1} - x^{(i_1}) \dots (\hat{x}^{i_n)} - x^{i_n)}) \rangle = \sum_{k=0}^{n} (-)^k \binom{n}{k} x^{(i_1} \dots x^{i_k} \langle \hat{x}^{i_{k+1}} \dots \hat{x}^{i_n)} \rangle,$$

- and their vector fields $X_{x^i}|_\Psi = \frac{1}{i\hbar} \hat{x}^i \Psi$ and

$$X_{G^{i_1 \dots i_n}} = \frac{1}{i\hbar} \sum_{k=0}^{n} (-)^k \binom{n}{k} \left(k \hat{x}^{(i_1} x^{i_2} \dots x^{i_k} \langle \hat{x}^{i_{k+1}} \dots \hat{x}^{i_n)} \rangle + x^{(i_1} \dots x^{i_k} \hat{x}^{i_{k+1}} \dots \hat{x}^{i_n)} \right) \Psi.$$

- Using that $\left[\hat{x}^i, \hat{x}^j \right] = i\hbar \epsilon^{ij}$, we can now prove that $\left\{ x^i, x^j \right\} = \epsilon^{ij}$,

$$\left\{ x^j, G^{i_1 \dots i_n} \right\} = \Omega(X_{x^j}, X_{G^{i_1 \dots i_n}}) = 0, \qquad \text{and}$$

$$\left\{ G^{a,n}, G^{b,m} \right\} = \sum_r \left[\left(\tfrac{\hbar}{2}\right)^{2r} K[a,b,m,n,r] G^{a+b-2r-1, m+n-4r-2} \right]$$

$$- \frac{b(n-a)}{m-b+1} G^{a,n-1} G^{b-1,m-1} + \frac{a(m-b)}{n-a+1} G^{b,m-1} G^{a-1,n-1}$$

where $G^{a,n} := G^{(p \dots pq \dots q)}$ with $a = \#p$ and $n = \#p + \#q$.

Quantum Evolution

- The dynamics of a quantum system has now become a classical dynamics of infinite degrees of freedom:

$$\dot{x}^i = \{x^i, H_Q\}$$
$$\dot{G}^{a,n} = \{G^{a,n}, H_Q\}$$

- Where H_Q is the Hamiltonian defined through

$$H_Q := \langle H(\hat{x}^i)\rangle_{Weyl} = \langle H(x^i + (\hat{x}^i - x^i))\rangle = \sum_{n=0,a=0}^{\infty,n} \frac{1}{n!}\binom{n}{a}\frac{\partial^n H(q,p)}{\partial p^i \partial q^{n-a}} G^{a,n}$$

- The set of equations is equivalent to Schrödinger equations of the system.

Effective Actions

- We say that the system of equations is effective when the purely quantum degrees of freedom $G^{i_1\cdots i_n}$ of the theory are sections on the fiber bundle and their dynamics occurs only on the base manifold:

$$\dot{G}^{i_1\ldots i_n} = \dot{x}^i \partial_{x^i} G^{i_1\ldots i_n}(x^j)$$

- Under certain circumstances the equations of motion can be derived from an action principle, from an Effective action.

- For the Anharmonic oscillator, by following the designed procedure we obtain

$$\Gamma_{\text{eff}}[q] = \int dt \left[\frac{1}{2} \left(m + \frac{\hbar (U''')^2}{2^5 m^2 \left(\omega^2 + \frac{U''}{m}\right)^{\frac{5}{2}}} \right) \dot{q}^2 - U - \frac{\hbar\omega}{2} \left(1 + \frac{U''}{m\omega^2}\right)^{\frac{1}{2}} \right]$$

to first order in the derivative expansion and first order in $\hbar$.

States with effective dynamics

- Given a solution to the equations $G^{i_1\cdots i_n}(q,p)$, the associated state (density matrix) $\hat\rho_U(q,p)$ can be reconstructed given that for $|\Psi_1\rangle$, $|\Psi_2\rangle$ it holds that

$$\int \frac{d^2z}{2\pi}\langle\Psi_1|e^{z\hat a^\dagger-\bar z\hat a}|\Psi_2\rangle Tr\{\hat\rho_U(q,p)e^{-z\hat a^\dagger+\bar z\hat a}\} = \langle\Psi_1|\hat\rho_U(q,p)|\Psi_2\rangle.$$

where $z=\frac{1}{\sqrt2}(z^q+iz^p)$, $\hat a=\frac{1}{\sqrt{2\hbar}}(\hat q+i\hat p)$ and

$$Tr\{\hat\rho_U(q,p)e^{-z\hat a^\dagger+\bar z\hat a}\}=e^{\frac{i}{\sqrt\hbar}(z^qp-z^pq)}\sum_{n=0,a=0}^{\infty,n}\frac{(-)^{n-a}i^n}{n!}\binom{n}{a}(z^q)^a(z^p)^{n-a}G^{a,n}(q,p).$$

For the anharmonic oscillator to $0th$ order in $\hbar$ and in coherent state basis, using $S^i=\delta^{ij}\epsilon_{jk}({\alpha'}^k-\alpha^k)+i(x^i-\frac{{\alpha'}^i+\alpha^i}{2})$, its matrix elements are

$$\langle\alpha|\hat\rho_U(q,p)|\alpha'\rangle=\frac{e^{-\frac{1}{2\hbar}(i{\alpha'}^i\alpha^j\epsilon_{ij}+|\alpha'-\alpha|^2)+\frac{1}{2\hbar}S^{i_1}S^{i_2}(\frac12\delta^{ij}+G^{ij})^{-1}_{i_1i_2}}}{\sqrt{\det(\frac12\delta^{ij}+G^{ij})}}.$$

Further considerations

- A purely geometrical approach to obtain Effective Actions has been developed which gives account of the already known results.
- The construction of dynamical coherent states has been achieved.
- Effective systems of equations for systems with no saddle point (Quantum cosmology) have already been constructed and a procedure to get their effective actions is under construction.
- The procedure might provide a more friendly framework to deal with issues as for example Quantization of constrained systems.
- A generalization of the present approach to obtain effective theories to LQC or even LQG is to be considered as a major goal.