RECOVERING SPACETIME TOPOLOGY FROM A CAUSAL SET.

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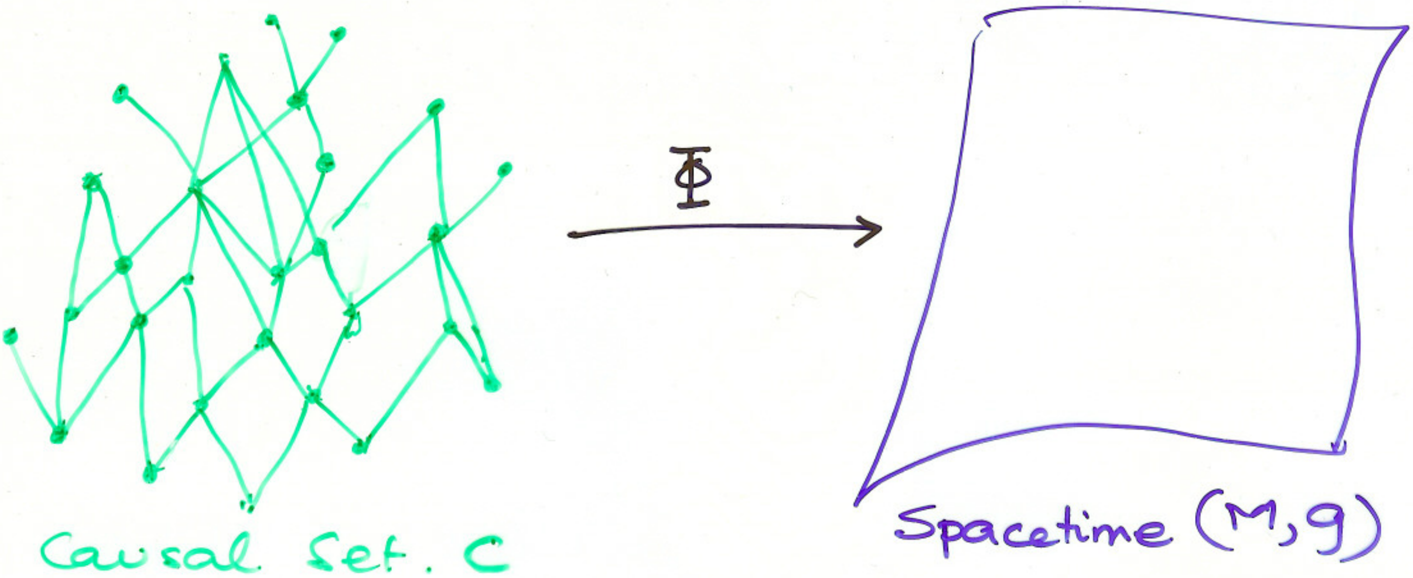
LOOPS '05.

THE CONTINUUM APPROXIMATION:



$\Phi : C \to M$ is a <u>FAITHFUL EMBEDDING</u> if:

(i) Partial order in C $\to$ causal order in $\Phi(C)$ w.r.t. g.

(ii) $\Phi(C)$ are randomly distributed in (M,g) at some density V_c^{-1}, with high probability.

Poisson distribution:

$$P_V(n) = \frac{e^{-V/V_c} (V/V_c)^n}{n!}$$

$$\langle n \rangle = \frac{V}{V_c} , \quad \sigma = \sqrt{V/V_c} \text{ deviation.}$$

$\to$ V: volume of spacetime interval

$\to$ n: # of "sprinkled" points in V.

IN CAUSAL SET THEORY, THERE IS

A FUNDAMENTAL DISCRETENESS $\Rightarrow V_c \geqslant V_P$.

$\therefore$ C CANNOT CONTAIN ALL THE CONTINUUM

TOPOLOGICAL INFORMATION IN M.

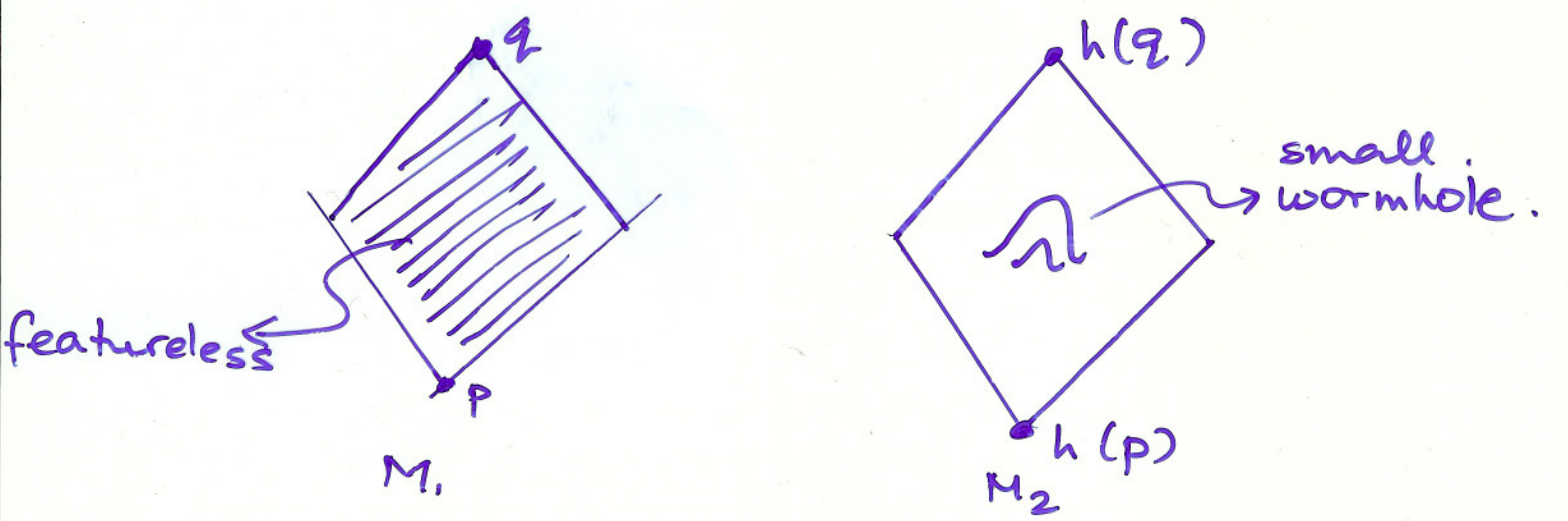
FUNDAMENTAL CONJECTURE : HAUPTVERMUTUNG.

$$\begin{tikzcd} & C \arrow[dl, "\Phi_1"'] \arrow[dr, "\Phi_2"] & \\ M_1 \arrow[rr, "h"'] & & M_2 \end{tikzcd}$$

$\Phi_2 = h \circ \Phi_1$

& h : approximate isometry from $g_1 \to g_2$.

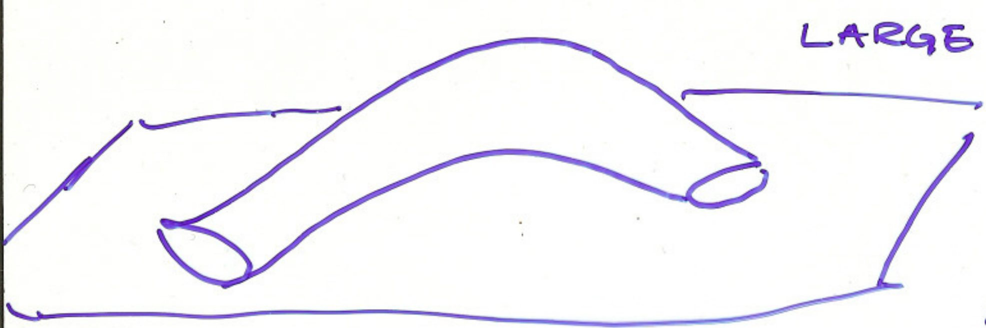
i.e. M_1 & M_2 ONLY DIFFER AT SCALES $\lesssim V_c$.



$M_1 \sim M_2$ if $vol(I(p,q)) \gtrsim V_c$.

SPACETIME "FOAM" IS $\therefore$ NOT RELEVANT TO A CAUSET.

TOPOLOGICAL STRUCTURES AT SCALES $\gg V_c$ **SHOULD** BE CAPTURED BY THE CAUSET.



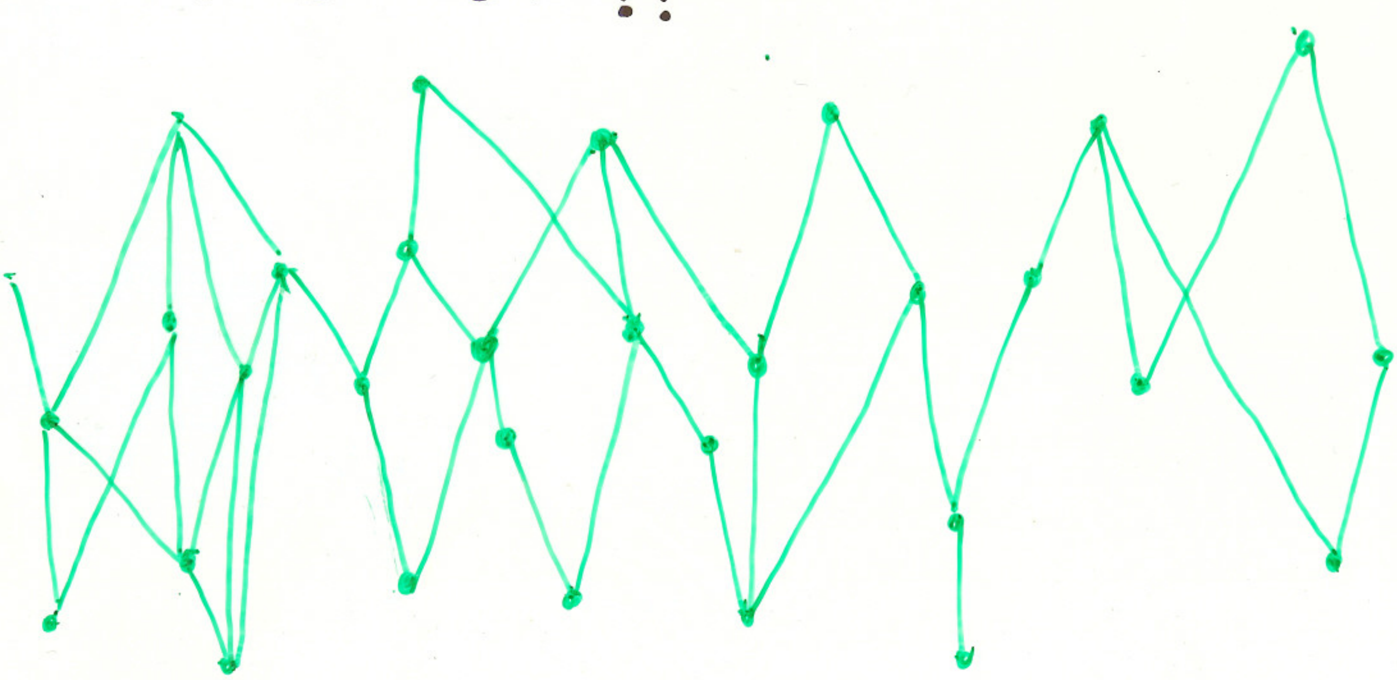
LARGE WORMHOLE.

AFFECTS THE CAUSAL STRUCTURE & $\therefore$ C SHOULD REFLECT THIS.

i.e. C SHOULD TELL US THAT SOMETHING WITH THE $\underbrace{\text{GROSS PROPERTIES}}$ OF A WORMHOLE MUST EXIST.

HOMOLOGY, HOMOTOPY, etc.

WHERE IS THIS INFORMATION HIDDEN IN A CAUSAL SET ??

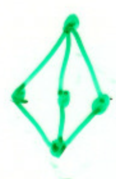


BECAUSE A CAUSAL SET IS NOT MERELY A DISCRETE SET OF POINTS, IT CAN BE ENDOWED WITH NON-TRIVIAL TOPOLOGICAL STRUCTURE.

- CHAIN COMPLEX :

k-element chain → k-simplex.

- INTERVAL TOPOLOGY.



- "FINITARY" TOPOLOGY. :

HOW ARE THESE RELATED TO THE TOPOLOGY OF THE CONTINUUM?.

- NEW PROPOSAL:

ANTI-CHAINS

A.

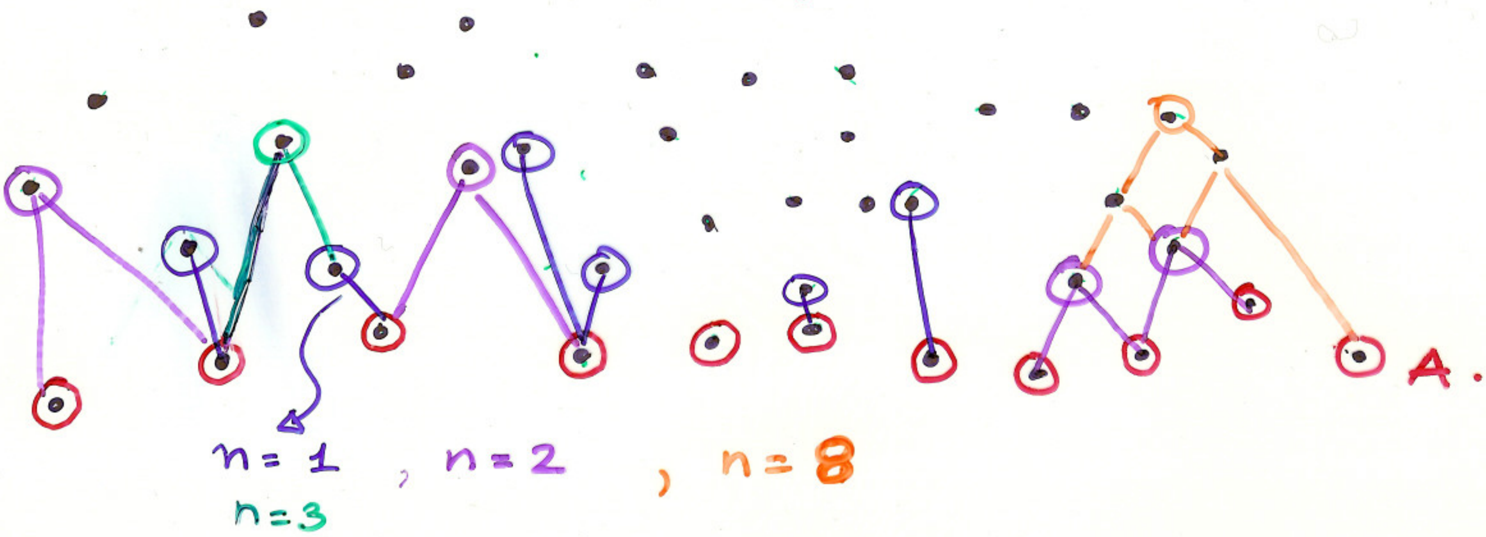
A : SET OF UNRELATED ELEMENTS IN C

~ ANALOGUE OF A SPATIAL HYPERSURFACE.

A ITSELF ONLY ALLOWS DISCRETE TOPOLOGY

THICKENED ANTICHAINS: A NERVE CONSTRUCTION

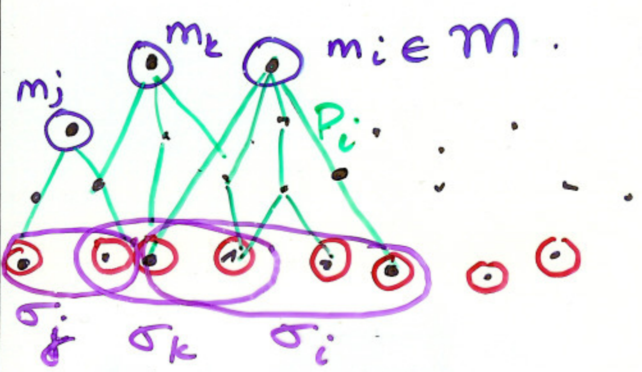
A : inextendible $\underbrace{\text{antichain}}_{\text{set of unrelated elements.}}$ in C.



"THICKEN" A :

$$\mathcal{J}_n(A) \equiv \{ p \in C \mid 0 < |\text{Past}(p) \cap \text{Fut}(A)| < n \}$$

$\mathcal{M}$: maximal / future most points of $\mathcal{J}_n(A)$



$P_i = \text{Past}(m_i) \cap \text{Fut}(A)$.

$\{P_i\}$ is a cover of $\mathcal{J}_n(A)$

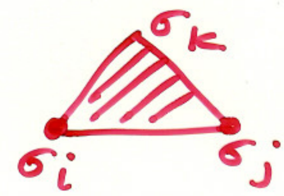
$\sigma_i = \text{Past}(m_i) \cap A$.

$\{\sigma_i\}$: cover of A.

$\sigma_i \longrightarrow$ vertex.

$\sigma_i \cap \sigma_j \neq \phi \longrightarrow$ 1-simplex:

$\sigma_i \cap \sigma_j \cap \sigma_k \neq \phi \longrightarrow$ 2-simplex



∴ $\sigma_n \equiv \{\sigma_i\} \longrightarrow N(\sigma_n)$] nerve of σ_n.

∴ CALCULATE HOMOLOGY GROUPS OF $N(\sigma_n)$, etc

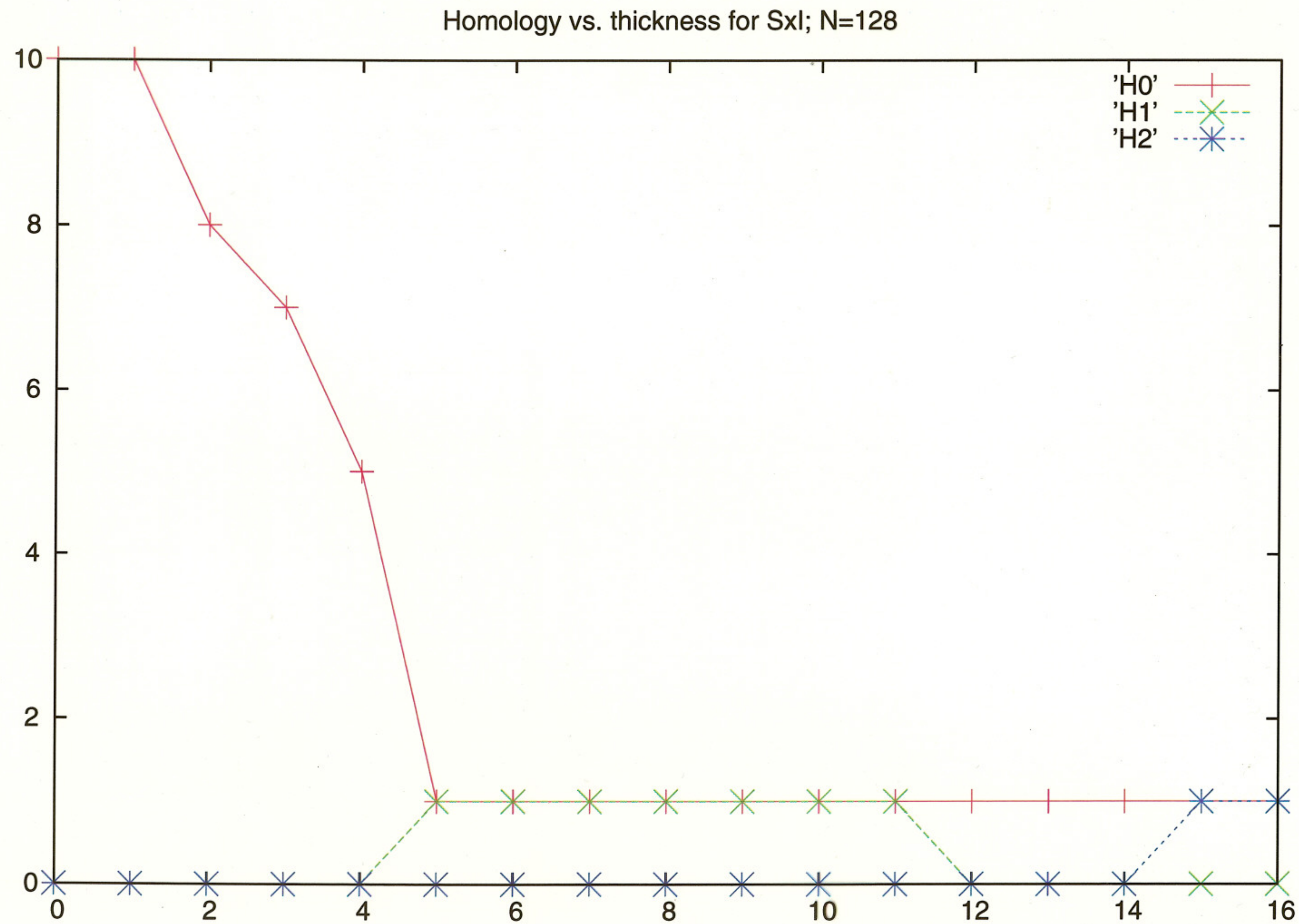
FOR EACH n, $N(\sigma_n)$ IS DIFFERENT. & CAPTURES LOCAL INFORMATION ABOUT A NBD. OF A in C.

· HOW IS THIS RELATED TO CONTINUUM HOMOLOGY?

DAVID'S NUMERICAL WORK + HOMOLOGY CALCULATOR

$\Rightarrow$ IT WORKS VERY WELL FOR $S^1 \times \mathbb{R}$ & $I \times \mathbb{R}$!

- FOR n VERY SMALL – LOTS OF DISCONNECTED PIECES IN $N_n(\sigma)$
- FOR n VERY LARGE – TOO MUCH "CONNECTEDNESS"!



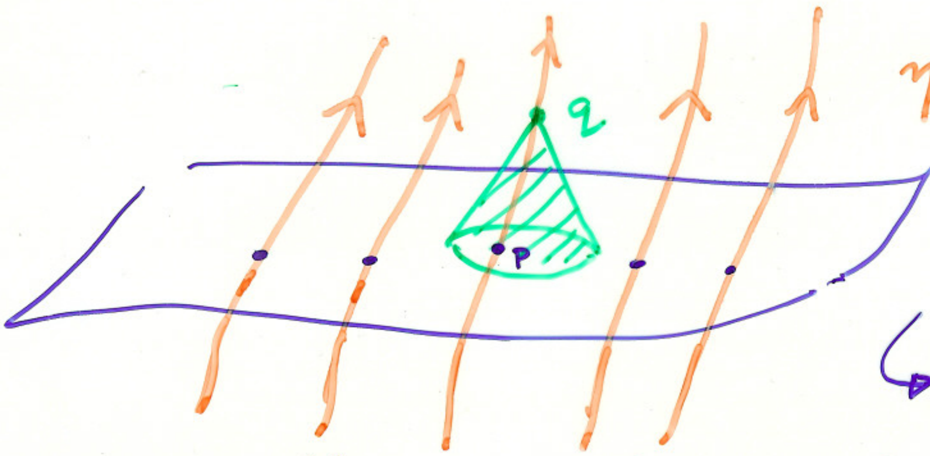
Homology vs. thickness for SxI; N=128
'H0'
'H1'
'H2'
0
2
4
6
8
10
12
14
16

(A) WHAT IS THE CONTINUUM ANALOGUE?

(B) HOW GOOD IS THE CAUSET-CONTINUUM CORRESPONDENCE FOR THE NERVES?

(A)

(M, g) : GLOBALLY HYPERBOLIC.



η^a : TIMELIKE VECTOR FIELD

Σ_0: CAUCHY SURFACE (ANALOGUE TO A).

$\boxed{\Sigma_0\text{: COMPACT}}$.

$\langle n \rangle = V/V_c$.

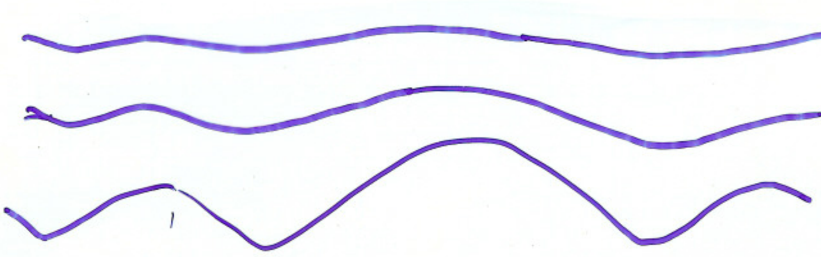
$\zeta(q) = \mathrm{vol}\left(I^+(\Sigma_0) \cap I^-(q)\right)$.

ζ : MONOTONICALLY INCREASES ALONG η^a.

& IS CONTINUOUS. (BECAUSE (M,g) IS CAUSALLY CONTINUOUS).

$\therefore$ Σ_ζ : LOCUS OF ALL PTS. WITH $\zeta(q) = \zeta$.

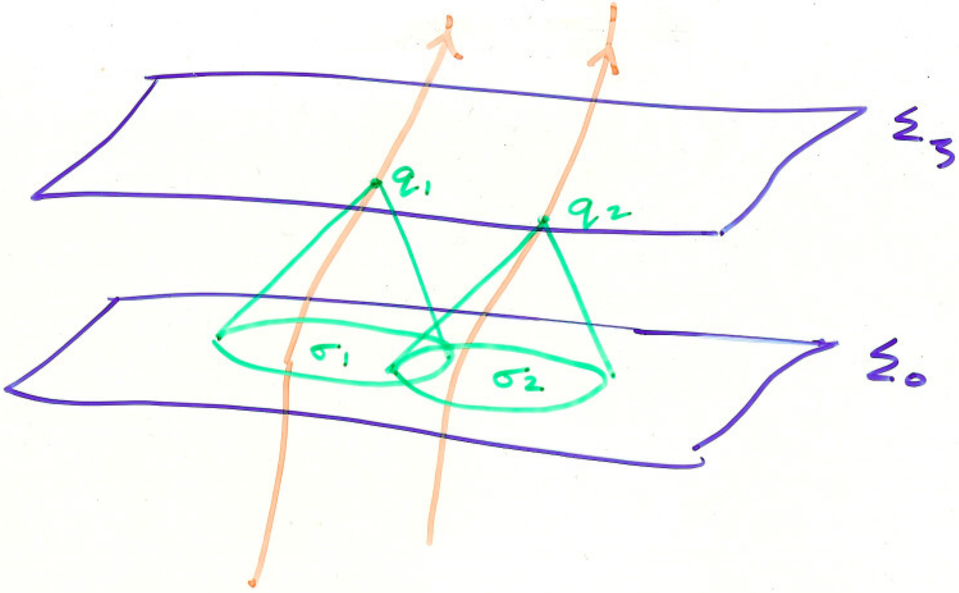
IS HOMEOMORPHIC TO Σ_0.



} "SMOOTHING" OUT OF Σ_0

Σ_0.

UNIFORMISATION OF EXTRINSIC CURVATURE



$\{\sigma(q)\}$: OPEN COVER OF Σ_0.

$\{\sigma(q_i)\}$: FINITE SUBCOVER. & $N(\sigma)$: NERVE.

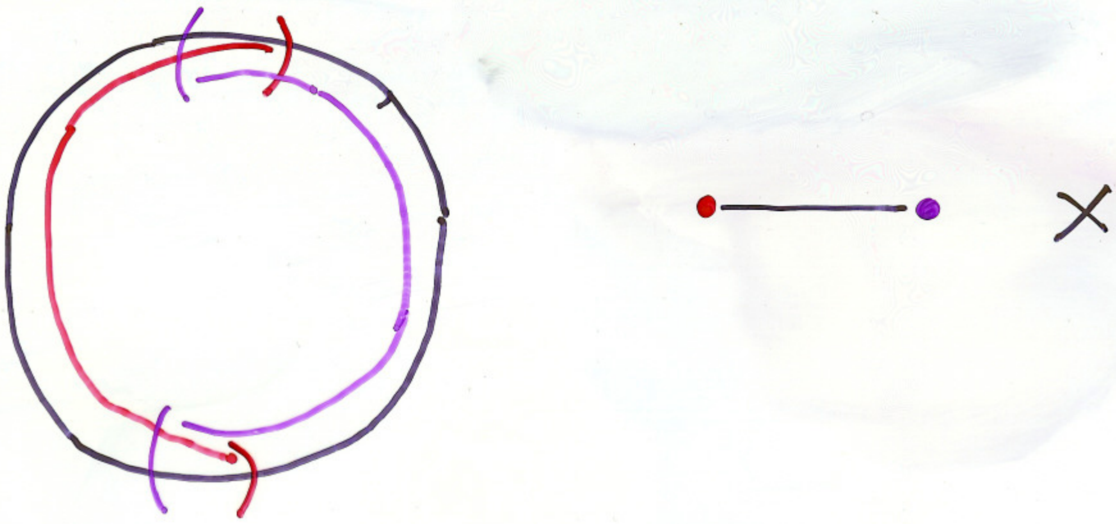
- ČECH COHOMOLOGY : USES $N(\sigma)$, BUT ONLY IN THE LIMIT OF ∞ REFINEMENT OF σ.

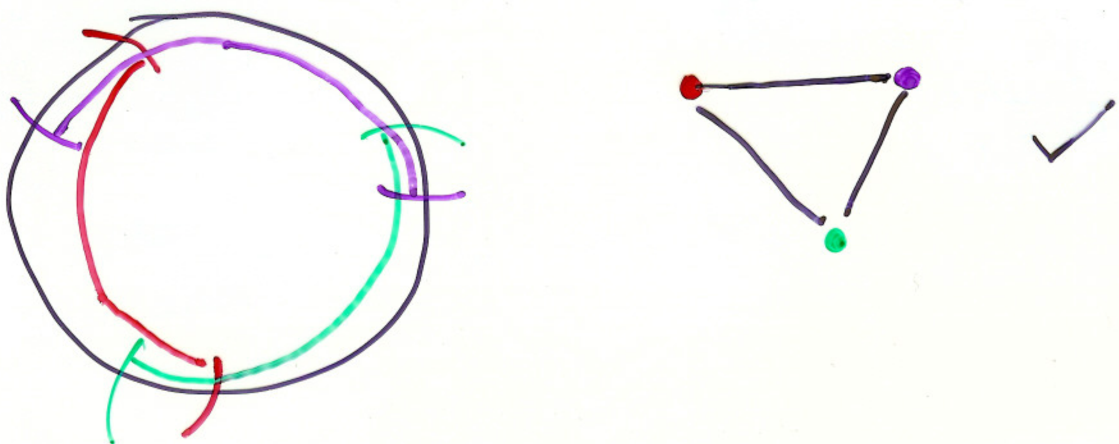
FUNDAMENTAL DISCRETENESS $\Rightarrow$ THIS CAN'T BE RELEVANT FOR THE CAUSAL SET.

SO HOW CAN THE HOMOLOGY OF $N(\sigma)$ BE RELATED TO Σ_0 IN GENERAL?

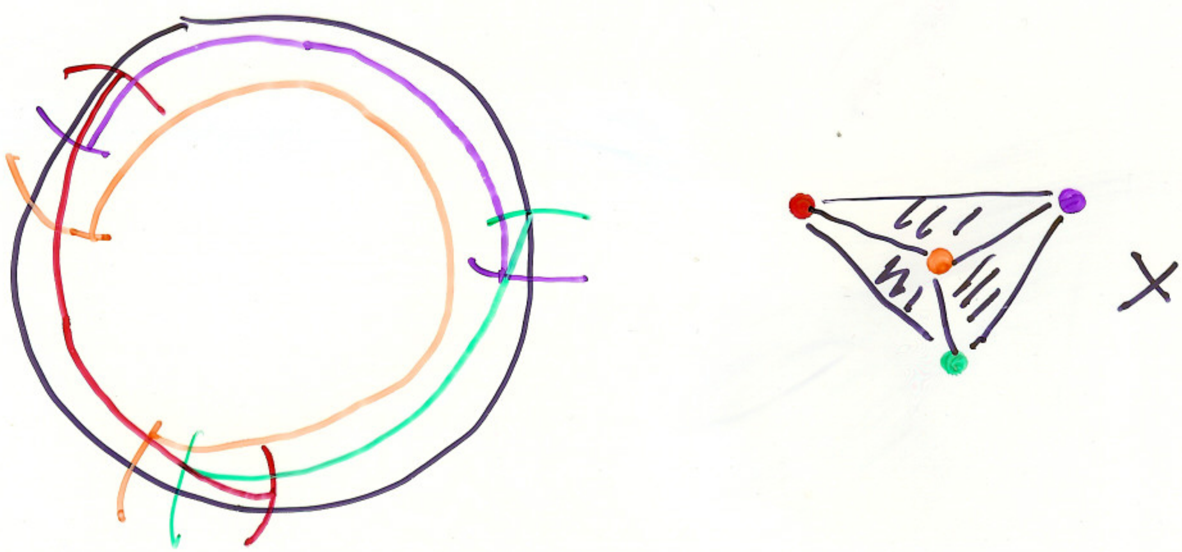
EXAMPLE : S^1.

$U_1 \cap U_2$





$U_4 \cap U_1$



SOMETIMES IT WORKS ...
& SOMETIMES IT DOESN'T !

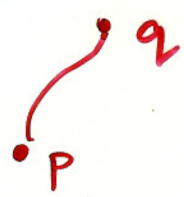
DE RHAM & WEIL:

LE NERF D'UN RECOUVREMENT CONVEXE DE V
A LE MÊME TYPE D'HOMOTOPIE QUE V.

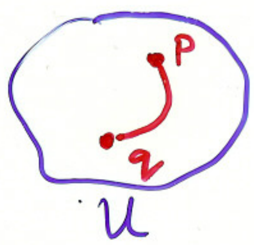
~ THE NERVE OF A "CONVEX COVER" OF Σ_0
IS HOMOTOPIC TO Σ_0.

CONVEX COVER:

(Σ, h) : RIEMMANIAN, THEN $\exists$ $\delta > 0$:



IF $d(p,q) < \delta$ THEN $\exists$ A UNIQUE GEODESIC FROM p to q.

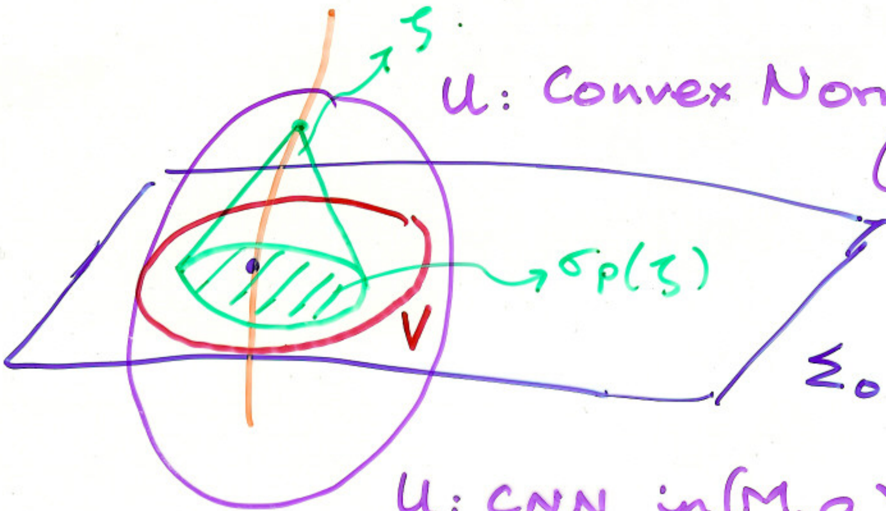


$\rightsquigarrow$ CONVEX SET IF EVERY GEODESIC OF ARC LENGTH $< \delta$ WITH END PTS IN $\mathcal{U}$ IS CONTAINED IN $\mathcal{U}$.

12.

SIMPLE COVER:

$\{U_i\}$: U_i : DIFFERENTIABLY CONTRACTIBLE
& $\bigcap_{i \in I} U_i$ ALSO " .



U: Convex Normal Neighbourhood (things "look like" Minkowski spacetime)

$\Sigma_0 \to$ FIND δ.

U: CNN in (M, g)

V: " in (Σ_0, h).

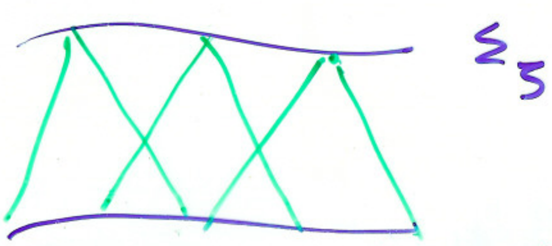
CHOOSE ξ SMALL ENOUGH THAT:

$\mathrm{Diam}(\sigma_p(\xi)) < \delta$, $\partial\sigma_p(\xi)$: +ve principal curvatures.

$\Rightarrow \sigma_p(\xi)$ IS CONVEX W.R.T. δ.

FOR EACH p FIND LARGEST ξ_p.

AND DEFINE $\boxed{\xi_0 = \inf_{p \in \Sigma_0} \xi_p.}$ "CONVEXITY VOLUME OF Σ_0".



$\xi < \xi_0$.

$\to$ OPEN CONVEX COVER OF Σ_0.

$\therefore$ GO TO A FINITE SUBCOVER $\tilde{\sigma}$

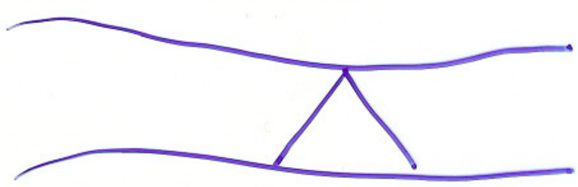
& DE-RHAM-WEIL THEOREM

$\Rightarrow N(\tilde{\sigma})$ HOMOTOPIC TO Σ_0 & $\therefore$ M.

(B) THE CORRESPONDENCE:

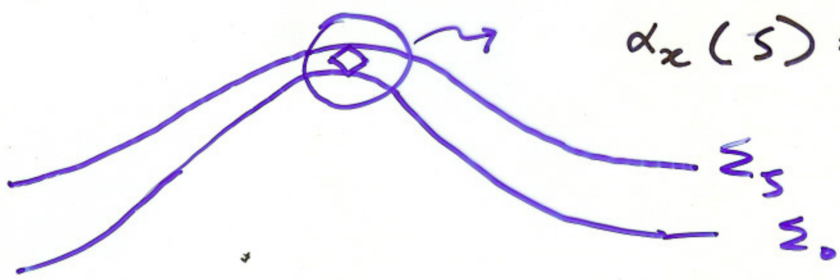
WHEN SHOULD WE EXPECT IT TO WORK?

(i) $\zeta_0 \gg V_c$



$\left.\begin{matrix}\Sigma_\zeta\\ \Sigma_0\end{matrix}\right\} M_\zeta \, , \quad \zeta_0 > \zeta \gg V_c .$

(ii) NO LOCALLY "THIN" REGIONS:



$$\alpha_x(\zeta) = \sup_{y \in \Sigma_\zeta} \mathrm{vol}(I(x,y))$$

$x \in \Sigma_0$

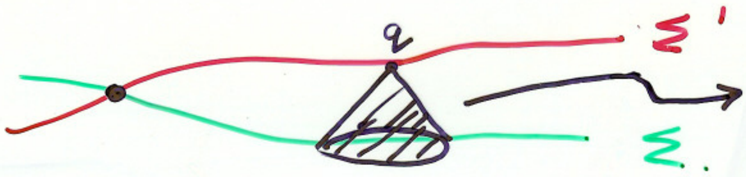
$\therefore \alpha_x(\zeta) \gg V_c .$

FOR SUFFICIENTLY HIGH BUT <u>FINITE</u> SPRINKLING DENSITY V_c^{-1} THIS CAN ALWAY BE ARRANGED.

$\Phi : C \to (M, g)$.

A: inextendible antichain in $\Phi(C) \subset M$.

GIVEN A $\exists$ ∞ # of Σ's CONTAINING A.



$I(\Sigma, q)$ IS SPACELIKE TO A.

SINCE A IS INEXTENDIBLE, THE REGION SPACELIKE TO IT MUST HAVE NO SPRINKLED POINTS.

$\therefore$ $I(\Sigma, q)$ MUST HAVE NO SPRINKLED POINTS.

$P_V(0) = e^{-V/V_c} \Rightarrow$ LARGE VOIDS ARE VERY IMPROBABLE

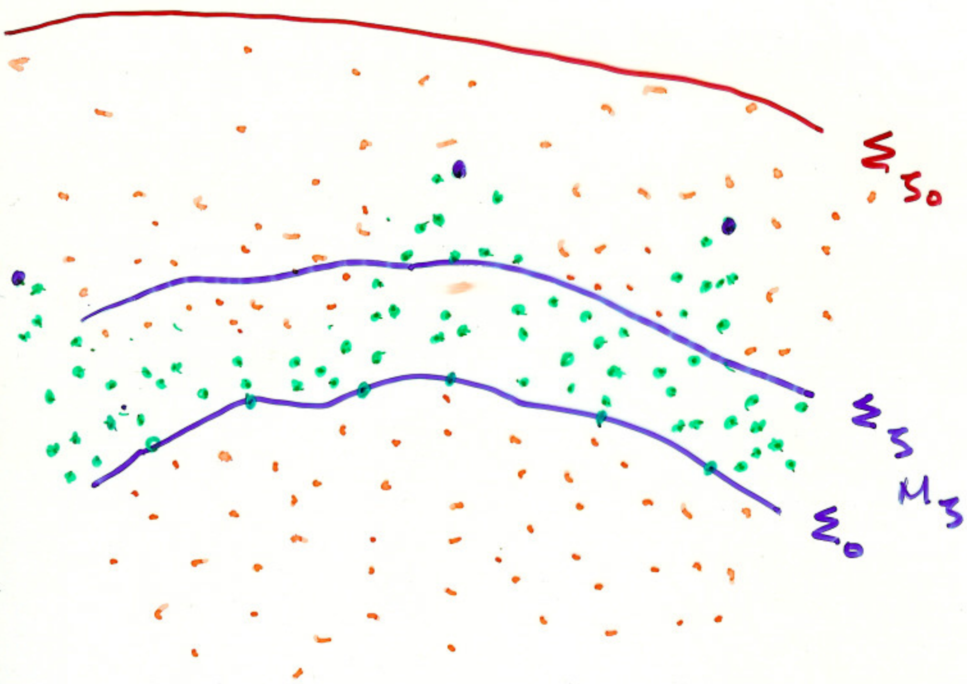
$\therefore$ WITH HIGH PROBABILITY,

$\mathrm{vol}[I(\Sigma, q)] \sim V_c$.

CHOOSE Σ_0 WITH THE LARGEST CONVEXITY RADIUS ζ_0.

$\mathcal{J}_n(A)$, $\zeta_0 \gg \zeta = nV_c$.

+ REQUIREMENT THAT M_ζ IS NOT TOO THIN.

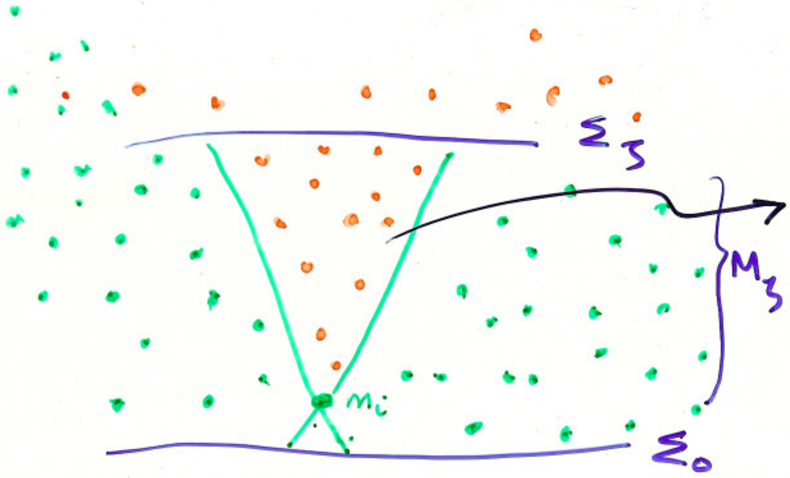


$\mathcal{M} = \{m_i\}$: SET OF MAXIMAL ELEMENTS.

$n(m_i) \leq n$.

WITH HIGH PROBABILITY $\zeta(m_i) < \zeta_0$.

CAN $\zeta(m_i) \not\leq V_c$?



ALL POINTS HERE HAVE $n(q) > n$ & $\zeta(q) < \zeta = nV_c$.

THIS REGION HAS VOLUME $\gg V_c$ BY ASSUMPTION.

$\Rightarrow$ THIS HAS VERY LOW PROBABILITY

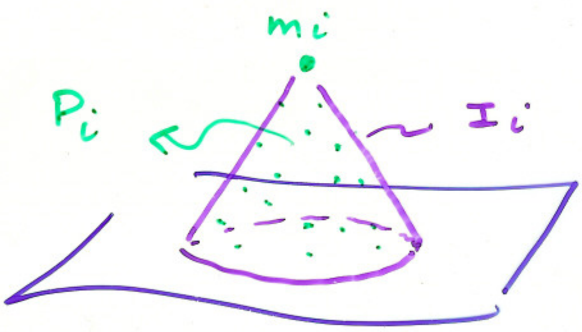
$\sim$ LARGE GAPS IN M_ζ OCCUR WITH V. LOW PROBABILITY.

"NO GAP LEMMA".

COMPARISON:

$P_i = \text{Past}(m_i) \cap \text{Fut}(A) \quad \subset \mathcal{J}_n(A)$] DISCRETE

$I_i = I^-(m_i) \cap I^+(\Sigma_0)$



$\mathcal{P} = \{P_i\}$ COVERS $\mathcal{J}_n(A)$

$\mathcal{I} = \{I_i\}$:

$N(\mathcal{I}) \simeq N(\sigma)$ CAN BE SHOWN.

$\therefore N(\mathcal{I}) \sim \Sigma_0$ (IF NO LARGE GAPS).

HOW ARE $N(\mathcal{P})$ & $N(\mathcal{I})$ RELATED?

VERTICES ARE $\{m_i\}$ & ARE THE SAME.

$N(\mathcal{P})$ IS A SUBCOMPLEX OF $N(\mathcal{I})$.

IF $I_i \cap I_j$ HAS VOL $\sim V_c$.

IT IS LIKELY THAT $P_i \cap P_j = \phi$.

HOMOLOGY OF A SUBCOMPLEX IS __NOT__

IN GENERAL $\simeq$ HOMOLOGY OF THE COMPLEX

..... UNLESS IT HAPPENS TO BE AN

ADEQUATE SUBCOMPLEX. .

(i) ALL NON-TRIVIAL CYCLES OF COMPLEX CORRESPONDS TO ONE IN THE SUBCOMPLEX AS WELL.

(ii) ANY TRIVIAL CYCLE IN THE COMPLEX IS ALSO TRIVIAL IN THE SUBCOMPLEX.

"NO GAP LEMMA" $\Rightarrow$ $N(\mathcal{P})$ IS AN ADEQUATE SUBCOMPLEX OF $N(\mathcal{J})$ WITH HIGH PROBABILITY.

$\therefore$ $N(\mathcal{P})$ IS HOMOLOGICALLY EQUIVALENT TO Σ_0, WITH HIGH PROBABILITY.

OPEN QUESTIONS.

- EXPLICIT CALCULATION OF PROBABILITIES.
  - NUMERICAL WORK.

- DOES IT LEND SUPPORT TO THE FUNDAMENTAL CONJECTURE?

$$\begin{tikzcd} & C \arrow[dl, "\phi_1"'] \arrow[dr, "\phi_2"] & \\ M_1 & & M_2 \end{tikzcd} \overset{?}{\Rightarrow} H_i(M_1) \simeq H_i(M_2).$$