Generally Covariant QFT

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Plan of talk:

I. GENERALLY COVARIANT QFT

II. SPIN AND STATISTICS, PCT ON CURVED SPACETIMES

III. WICK PRODUCTS AND TIME-ORDERED PRODUCTS: reducing the renormalization ambiguity

I. GENERALLY COVARIANT QFT

Three essential principles — for QFT

- **Principle of locality**

  The **dynamical laws** and **(algebraic) relations** of the **observables** of a QFT on a fixed background spacetime should be **locally determined**, without dependence on the global structure of spacetime.

- **Principle of general covariance**

  The **dynamical laws** and **relations** of the **observables** of a QFT are **equivalent on isometric spacetimes**.

- **Principle of dynamical determination of spacetime**

  Spacetime not fixed but determined by Einstein's equations — thus there should be QFTs of the "same type" on all spacetimes [which can occur as solutions to Einstein's equations]

N.B. While the relations between the **observables** are required to be **local** ("finite propagation speed"), there are typically **nonlocal correlations** at the level of the **states** $\longrightarrow$ non-local quantum effects (EPR, Bell-correlations,...)

Combination of these 3 principles into the

generally covariant locality principle

To each spacetime $(M, \boldsymbol{g})$, associate a QFT: $(M, \boldsymbol{g}) \mapsto \phi_{(M,\boldsymbol{g})}$ which is of the **same type** on all spacetimes, i.e.

$\boxed{\text{the observables of the QFT on } (M, \boldsymbol{g})}$ and

$\boxed{\text{the observables of the QFT on } (M', \boldsymbol{g}')}$

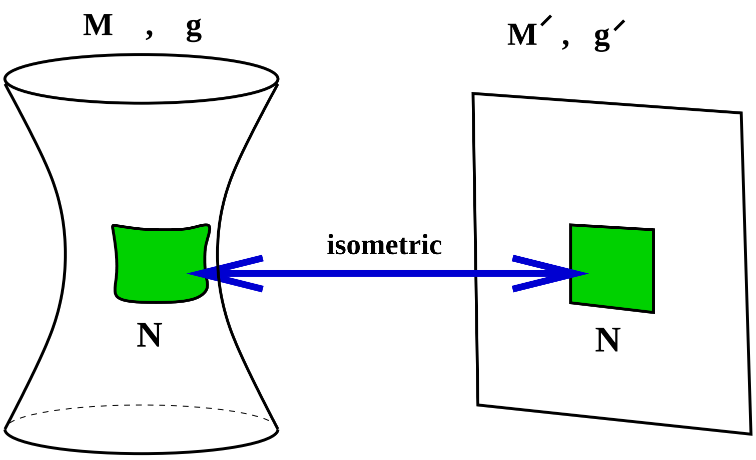
should have the **same structure (algebraic relations, dynamics, field content) on isometric subregions**.

In other words:

$\mathcal{A}(M, \boldsymbol{g}) = \boxed{\text{*-algebra of observables generated by } \phi_{(M,\boldsymbol{g})}}$

if $(M, \boldsymbol{g})$ and $(M', \boldsymbol{g}')$ are **isometric** on a subregion N, then there should be an **isomorphism**

$$\mathcal{A}(N, \boldsymbol{g}) \xrightarrow{\alpha} \mathcal{A}(N, \boldsymbol{g}')$$



Local, generally covariant (functorial) QFT (BFV, 2003)

The mathematical content of the generally covariant locality principle is best given using the language of categories and functors. We need two categories:

$\mathfrak{Man}$:

Objects: four-dimensional, globally hyperbolic spacetimes $(M, \boldsymbol{g})$ which are oriented and time-oriented.

$$\mathcal{M} = (M, \boldsymbol{g}, \epsilon_{abcd}, T)$$

Morphisms: $\psi \in \hom_{\mathfrak{Man}}(\mathcal{M}_1, \mathcal{M}_2)$ if

$$\psi : \mathcal{M}_1 \longrightarrow \mathcal{M}_2$$

- is an isometric embedding, $\psi_* g_1 = g_2$
- preserves orientation and time-orientation
- is causally regular, ie $\psi(\mathcal{M}_1)$ is a globally hyperbolic sub-spacetime of $\mathcal{M}_2$,

$$J^+_{\mathcal{M}_2}(\psi(p)) \cap J^-_{\mathcal{M}_2}(\psi(q)) \subset \psi(\mathcal{M}_1)\,, \quad \forall\, p, q \in \mathcal{M}_1$$

$\mathfrak{Alg}$:

Objects: $*$-algebras (favorably: C^*-algebras) possessing unit elements

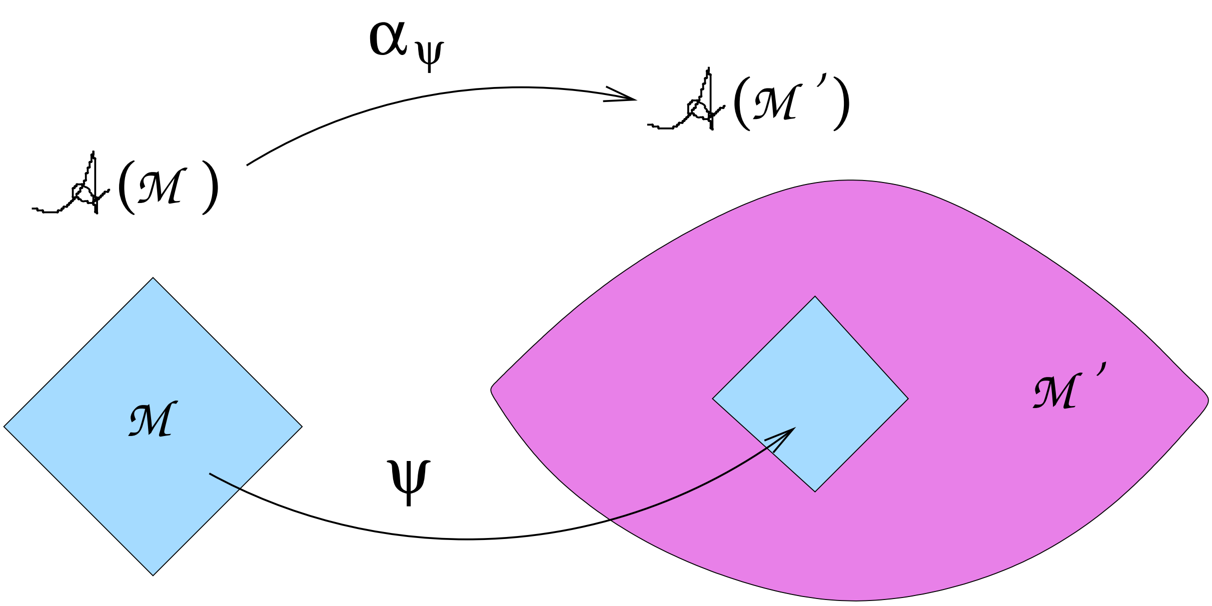
Morphisms: $\alpha \in \hom_{\mathfrak{Alg}}(\mathcal{A}_1, \mathcal{A}_2)$ if $\alpha : \mathcal{A}_1 \longrightarrow \mathcal{A}_2$ is a linear, faithful (injective) unit-preserving $*$-homomorphism.

A **QFT fulfilling the generally covariant locality principle** (shorter: **local covariant QFT**) is a covariant functor $\mathscr{A}$ between the two categories $\mathfrak{Man}$ and $\mathfrak{Alg}$, i.e., writing α_ψ for $\mathscr{A}(\psi)$, in typical diagramatic form:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\psi} & \mathcal{M}' \\ {\scriptstyle\mathscr{A}}\downarrow & & \downarrow{\scriptstyle\mathscr{A}} \\ \mathscr{A}(\mathcal{M}) & \xrightarrow{\alpha_\psi} & \mathscr{A}(\mathcal{M}') \end{array}$$

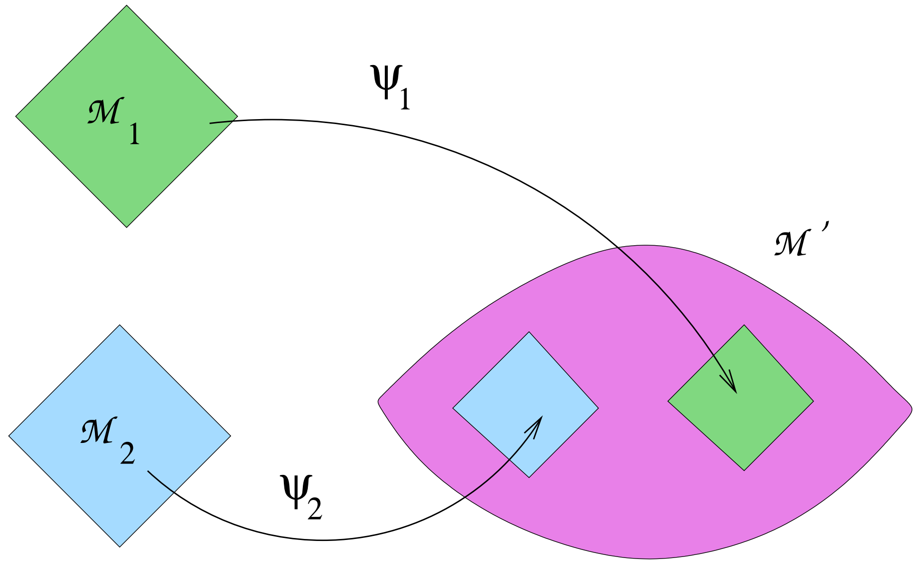
Covariance means:

$$\alpha_{\psi'} \circ \alpha_\psi = \alpha_{\psi' \circ \psi}\,, \quad \alpha_{\mathrm{id}_{\mathcal{M}}} = \mathrm{id}_{\mathscr{A}(\mathcal{M})}\,,$$



A loc. cov. QFT $\mathscr{A}$ is called **causal** if the following holds:
if $\psi_1(\mathcal{M}_1)$ and $\psi_2(\mathcal{M}_2)$ are causally separated in $\mathcal{M}'$,

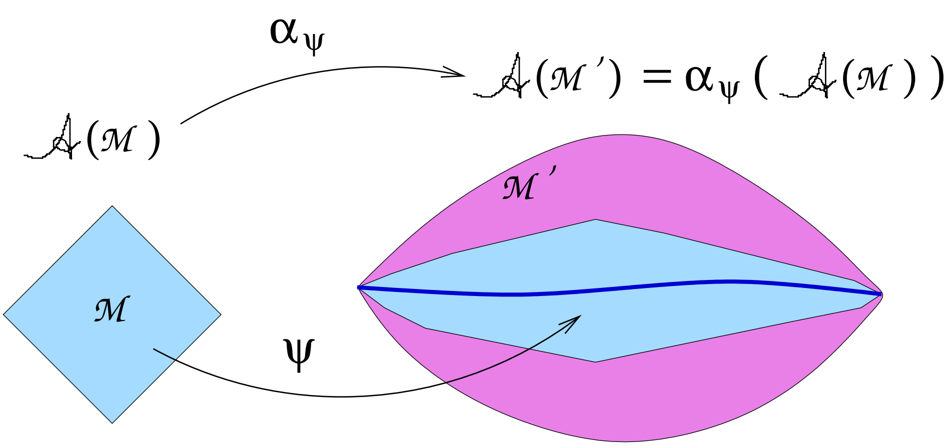
one has $$\left[\alpha_{\psi_1}(\mathscr{A}(\mathcal{M}_1)), \alpha_{\psi_2}(\mathscr{A}(\mathcal{M}_2))\right] = \{0\}\,,$$



$\mathscr{A}$ obeys the **time-slice axiom** if

$$\alpha_\psi(\mathscr{A}(\mathcal{M})) = \mathscr{A}(\mathcal{M}')$$

when $\psi(\mathcal{M})$ contains a Cauchy-surface for $\mathcal{M}'$:



Example : Define for all $\mathcal{M} \in \mathrm{Obj}(\mathfrak{Man})$ the C^*-algebraic variant of the free scalar field; i.e. (in formal notation) define

$$W_{\mathcal{M}}(f) = \mathrm{e}^{i\phi_{\mathcal{M}}(f)}\,, \quad f \in C_0^\infty(M, \mathbb{R})\,,$$

then the $W_{\mathcal{M}}(f)$ obey the **"exponentiated CCR"** (Weyl-relations)

$$\begin{aligned} W_{\mathcal{M}}(f)W_{\mathcal{M}}(h) &= \mathrm{e}^{-iE(f\otimes h)}W_{\mathcal{M}}(f)W_{\mathcal{M}}(h)\,, \\ W_{\mathcal{M}}(f)^* &= W_{\mathcal{M}}(-f) \\ W_{\mathcal{M}}(f + (\nabla^\mu\nabla_\mu + m^2)h) &= W_{\mathcal{M}}(f)\,. \end{aligned}$$

Then one obtaines a **local covariant QFT** via

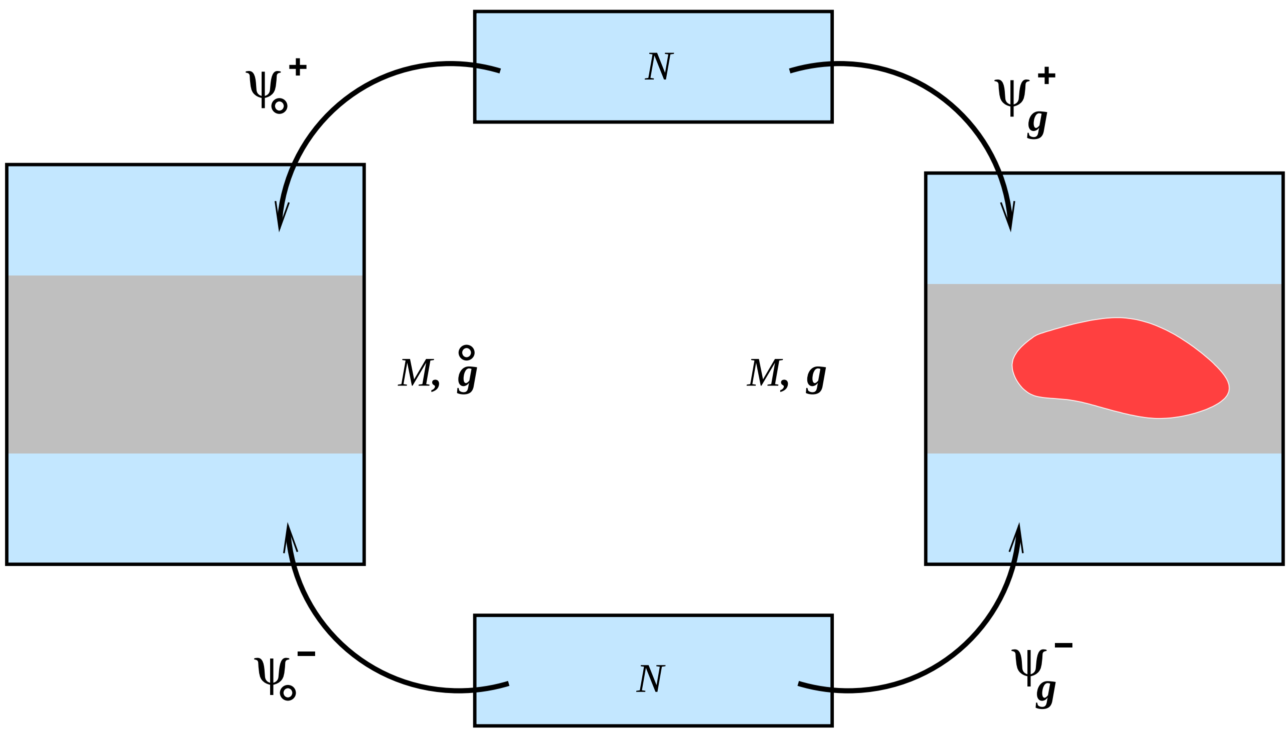
$$\begin{aligned} \mathscr{A}(\mathcal{M}) &= C^*\text{-algebra generated by all } W_{\mathcal{M}}(f) \\ \alpha_\psi(W_{\mathcal{M}}(f)) &= W_{\mathcal{M}'}(\psi_* f) \end{aligned}$$

The locally covariant QFT obtained in this way is also

- **causal** and
- fulfills the **time-slice axiom**.

Dynamics

Suppose we have a globally hyperbolic spacetime $(M, \mathring{\boldsymbol{g}})$ and we vary the metric $\mathring{\boldsymbol{g}}$ to a metric $\boldsymbol{g}$ in a fixed neighbourhood of a Cauchy-surface:



Assuming that $\mathscr{A}$ fulfills the **time-slice axiom**, we can construct a C^*-algebra automorphism

$$\beta_{\boldsymbol{g}} = \alpha_{\psi_\circ^-} \circ \alpha_{\psi_{\boldsymbol{g}}^-}^{-1} \circ \alpha_{\psi_{\boldsymbol{g}}^+} \circ \alpha_{\psi_\circ^+}^{-1} \,,$$

called **relative Cauchy-evolution**, which results from the functorial properties of $\mathscr{A}$ via the diagram:

$$\begin{tikzcd}
 & \mathscr{A}(N, \boldsymbol{g}) \arrow[dl, "\alpha_{\psi_\circ^+}"'] \arrow[dr, "\alpha_{\psi_{\boldsymbol{g}}^+}"] & \\
\mathscr{A}(M, \mathring{\boldsymbol{g}}) & & \mathscr{A}(M, \boldsymbol{g}) \\
 & \mathscr{A}(N, \boldsymbol{g}) \arrow[ul, "\alpha_{\psi_\circ^-}"] \arrow[ur, "\alpha_{\psi_{\boldsymbol{g}}^-}"'] &
\end{tikzcd}$$

Now assume that there is a Hilbert-space reprn $(\pi, \mathcal{H})$ of $\mathscr{A}(M, \mathring{\boldsymbol{g}})$ and that the functional derivative

$$\frac{\delta}{\delta \boldsymbol{g}_{\mu\nu}(x)} \pi(\beta_{\boldsymbol{g}} B)$$

can be formed for all B in a dense sub-algebra $\mathcal{B} \subset \mathscr{A}(M, \mathring{\boldsymbol{g}})$ in the sense of quadratic forms on a suitable dense domain $\mathcal{V} \subset \mathcal{H}$.

This quantity should have the significance of an **energy-momentum tensor**. In fact:

Theorem

Under these assumptions:

$$\nabla_\mu \frac{\delta}{\delta \boldsymbol{g}_{\mu\nu}(x)} \pi(\beta_{\boldsymbol{g}}(B)) = 0\,.$$

For the case $\mathscr{A}$ = free scalar field, it holds in the representations induced by Hadamard states on $\mathscr{A}(M, \mathring{\boldsymbol{g}})$ that

$$\frac{\delta}{\delta \boldsymbol{g}_{\mu\nu}(x)} \beta_{\boldsymbol{g}}(W(f)) = -\frac{i}{2}[T^{\mu\nu}(x), W(f)]$$

Remarks

(a) Non-existence of invariant states

Let $\mathscr{A}$ be a local covariant QFT.

A **state** is a family $\boldsymbol{\omega} = \{\omega_{\mathcal{M}}\}$, where

$$\omega_{\mathcal{M}} \quad \text{is a state on} \quad \mathscr{A}(\mathcal{M})\,.$$

A state is called **(diff-) invariant** if

$$\omega_{\mathcal{M}'} \circ \alpha_\psi = \omega_{\mathcal{M}}$$

for all $\psi : \mathcal{M} \longrightarrow \mathcal{M}'$.

In general, there are no such states!

E.g. for free fields, there exists **no** invariant state $\{\omega_{\mathcal{M}}\}$ so that the $\omega_{\mathcal{M}}$ are Hadamard states:

Existence amounts to $\beta_{\boldsymbol{g}} = \text{Identity} \Leftrightarrow T^{\mu\nu}(x) = 0$

(b) Tensorial structure and causality (K. Fredenhagen)

$\mathfrak{Man}$ is a tensor category with $\mathcal{M}_1 \amalg \mathcal{M}_2 =$ disjoint union
$\mathfrak{Alg}$ has "natural" structure of tensor category

Causality of a local cavariant QFT $\mathscr{A}$ can be expressed as saying that $\mathscr{A}$ is a **tensor functor**:

$$\mathscr{A}(\mathcal{M}_1 \amalg \mathcal{M}_2) \simeq \mathscr{A}(\mathcal{M}_1) \otimes \mathscr{A}(\mathcal{M}_2)$$

(c) Local covariant QFT in arbitrary dimensions

(R.V., C.J. Fewster)

$\widetilde{\mathfrak{Man}}$ = category of all finite-dim (≥ 2) gh spacetimes as objects, with same morphisms as before

For $\boldsymbol{\Sigma} = (\Sigma, \boldsymbol{h})$ = compact, connected Riemannian manifold, define a functor on $\widetilde{\mathfrak{Man}}$:

$$\mathcal{S}_{\boldsymbol{\Sigma}} : (M, \boldsymbol{g}) \mapsto (M \times \Sigma, \boldsymbol{g} \oplus (-\boldsymbol{h}))$$

$$\sigma_{\boldsymbol{\Sigma}} : \psi \mapsto \psi \times \mathrm{id}_\Sigma$$

Let $\mathscr{A} : \widetilde{\mathfrak{Man}} \longrightarrow \mathfrak{Alg}$ be a loc. cov. QFT functor as before.
We say that it describes the **same theory in all dimensions** if:
For all $\boldsymbol{\Sigma}$ there is a natural transformation $\gamma^{[\boldsymbol{\Sigma}]}$ between $\mathscr{A}$ and $\mathscr{A} \circ \mathcal{S}_{\boldsymbol{\Sigma}}$:

$$\begin{array}{ccc}
\mathscr{A}(\mathcal{M}) & \xrightarrow{\gamma^{[\boldsymbol{\Sigma}]}_{\mathcal{M}}} & \mathscr{A} \circ \mathcal{S}_{\boldsymbol{\Sigma}}(\mathcal{M}) \\
{\scriptstyle \alpha_\psi}\big\downarrow & & \big\downarrow{\scriptstyle \alpha_{\sigma_{\boldsymbol{\Sigma}} \circ \psi}} \\
\mathscr{A}(\mathcal{M}') & \xrightarrow{\gamma^{[\boldsymbol{\Sigma}]}_{\mathcal{M}'}} & \mathscr{A} \circ \mathcal{S}_{\boldsymbol{\Sigma}}(\mathcal{M}')
\end{array}$$

and

$$\gamma^{[\boldsymbol{\Sigma}]} \circ \gamma^{[\boldsymbol{\Sigma}']} = \gamma^{[\boldsymbol{\Sigma} \oplus \boldsymbol{\Sigma}']}$$

II. SPIN AND STATISTICS, PCT

For a quantum field ϕ on **Minkowski spacetime**, **Wightman's axioms** imply

- If ϕ has spin $n + \frac{1}{2} \quad (n \in \mathbb{N}_0)$
  then ϕ must be Fermionic
  (fulfills spacelike anti-commutativity)

- If ϕ has spin $n \quad (n \in \mathbb{N}_0)$
  then ϕ must be Bosonic
  (fulfills spacelike commutativity)

- There is an anti-unitary operator V_{pct} so that

$$V_{pct}\phi(f)V_{pct}^{-1} = i^{\kappa}\phi(\overline{f \circ PCT})$$

  where $\quad PCT : x \mapsto -x$,
  κ depends on spinor degree of ϕ

Aim: We would like to have similar theorems on curved spacetime at comparable level of generality,

$\sharp$ <u>without</u> specific assumptions on the spacetime (symmetries)

$\sharp$ <u>without</u> specific assumptions on the QFT (special models)

Quantum (spinor) fields

Let

$\mathcal{M} = ((M,g), S(M,g), \sigma)$ be a glob. hyp. spacetime with spin-structure,

ρ a finite-dim. irred. rep- of SL$(2,\mathbb{C})$ on V_ρ,

$\mathcal{V}_\rho(\mathcal{M})$ the corresp. associated vector bundle.

Then

$$\Phi_{\mathcal{M}} = (\phi_{\mathcal{M}}, \mathcal{D}, \mathcal{H}) \quad \text{is called a } \textbf{quantum field on } \mathcal{M} \text{ if:}$$

1) $\mathcal{H}$ = Hilbertspace, $\mathcal{D} \subset \mathcal{H}$ dense linear subspace

2) $\Gamma_0(\mathcal{V}_\rho(\mathcal{M})) \ni f \mapsto \phi_{\mathcal{M}}(f)$ linear,
   $\phi_{\mathcal{M}}(f)$ closable operator on $\mathcal{D}$,
   $\mathcal{D}$ invariant under all $\phi_{\mathcal{M}}(f)$

3) $\Gamma_0(\mathcal{V}_\rho(\mathcal{M})) \ni f \mapsto \langle \chi, \phi_{\mathcal{M}}(f)\chi' \rangle$
   is continuous, $\forall\ \chi, \chi' \in \mathcal{D}$

4) There are cyclic vectors $\chi \in \mathcal{D}$

Generally covariant QFT on spacetimes with spin-structure

In the spirit of a functorial description, we need a new category:

$\mathfrak{ManSp}$: The class of objects is formed by four-dim., globally hyperbolic spacetimes with spin-structure, each denoted

$$\mathcal{M} = ((M, \boldsymbol{g}), S(M, \boldsymbol{g}), \sigma, \epsilon_{abcd}, T)$$

Morphisms are the isometric embeddings preserving the spin-structure (up to equivalence):

$\boldsymbol{\Theta} = (\Theta, \vartheta)$ is called a local isomorphism between $\mathcal{M}_1$ and $\mathcal{M}_2$, denoted $\mathcal{M}_1 \xrightarrow{\boldsymbol{\Theta}} \mathcal{M}_2$, if:

(a) $(M_1, \boldsymbol{g}_1) \xrightarrow{\vartheta} (M_2, \boldsymbol{g}_2)$ is a morphism of $\mathfrak{Man}$

(b) θ lifts to a local spin-structure isomorphism

$$\Theta : S_1(M_1, \boldsymbol{g}_1) \to S_2(M_2, \boldsymbol{g}_2)$$

Note: ϑ preserves the orientation ϵ_{abcd} and the time-orientation T.

$\overline{\mathcal{M}}$ is the copy of $\mathcal{M}$ with the **reversed time-orientation**, $T \mapsto -T$.

Generally covariant QFT

A **generally covariant QFT of spin-reprn. type ρ** is a covariant functor $\mathscr{F}$ between $\mathfrak{ManSp}$ and $\mathfrak{Alg}$,

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\boldsymbol{\Theta}} & \mathcal{M}' \\ {\scriptstyle \mathscr{F}}\downarrow & & \downarrow{\scriptstyle \mathscr{F}} \\ \mathscr{F}(\mathcal{M}) & \xrightarrow{\alpha_{\boldsymbol{\Theta}}} & \mathscr{F}(\mathcal{M}') \end{array}$$

where, moreover, there is a family of quantum fields

$$\Phi_{\mathcal{M}} \quad \text{on} \quad \mathcal{M} \in \mathrm{Obj}(\mathfrak{ManSp})$$

so that each $\Phi_{\mathcal{M}}$ generates the algebra $\mathscr{F}(\mathbf{M})$

Remark : The $\mathscr{F}(\mathbf{M})$ are actually von Neumann algebras, and the action of $\alpha_{\boldsymbol{\Theta}}$ must be interpreted in a sense differing from the C^*-case before; but the formal structure of the functorial properties remains.

Spin-Statistics on manifolds (R.V., 2001)

Theorem

Let $\mathscr{F}$ be a generally covariant QFT of spin-reprn. type ρ.

Assume also:

$\sharp$ $\mathscr{F}$ fulfills the **time-slice axiom**,

$\sharp$ $\Phi_{\mathcal{M}_0}$ on $\mathcal{M}_0$ = Minkowski spacetime fulfills the **Wightman axioms.**

(I) Let ρ be half-integer.

Suppose: There is some $\mathcal{M} \in \mathrm{Obj}(\mathfrak{ManSp})$ and a pair of causally separated regions $O_1, O_2 \subset \mathcal{M}$ so that for all f_j with supp$f_j \subset O_j$,

$$\phi_{\mathcal{M}}(f_1) \text{ commutes strongly with } \phi_{\mathcal{M}}(f_2)\,.$$

Then : $\phi_{\mathcal{M}'}(f) = c_f \cdot 1 \;\; \forall\, \mathcal{M}' \in \mathrm{Obj}(\mathfrak{ManSp})$

(II) Let ρ be integer.

Suppose: There is some $\mathcal{M} \in \mathrm{Obj}(\mathfrak{ManSp})$ and for each pair of causally separated regions $O_1, O_2 \subset \mathcal{M}$ a pair of test-tensors f_1, f_2, supp $f_j \subset O_j$, with $\phi_{\mathcal{M}}(f_j) \neq 0$ and

$$\phi_{\mathcal{M}}(f_1)\phi_{\mathcal{M}}(f_2) + \phi_{\mathcal{M}}(f_2)\phi_{\mathcal{M}}(f_1) = 0\,.$$

Then : $\phi_{\mathcal{M}'}(f) = c_f \cdot 1 \;\; \forall\, \mathcal{M}' \in \mathrm{Obj}(\mathfrak{ManSp})$

9) Examples

1. Scalar Klein-Gordon field:
   $V_\rho = \mathbb{R}$, $\rho = D^{(0,0)}$,
   field eqn $\quad (\nabla^a \nabla_a + m^2)\phi = 0 \ (m \geq 0)$

2. Proca field:
   $V_\rho = \mathbb{R}^4$, $\rho = D^{(1,1)}$,
   field eqn $\quad (*d * d + m^2)\phi = 0 \ (m > 0)$

3. Majorana-Dirac field:
   $V_\rho = \mathbb{C}^4$, $\rho = D^{(1,0)} \oplus D^{(0,1)}$,
   field eqn $\quad (\nabla\!\!\!/ \ + im)\phi = 0 \ (m \geq 0)$

In cases 1. and 2., the quantum fields fulfil CCRs in exponentiated Weyl-form.

In case 3. the quantum fields fulfil CARs.

They are uniquely constructed from the field equations.

In all cases, the fields are represented on Hilbert-space representations induced by quasifree states fulfilling the microlocal spectrum condition **μSC**

PCT for loc. cov. QFTs admitting OPEs (Hollands, 2004)

OPE at $x \in \mathcal{M}$ for QFT $\Phi_{\mathcal{M}}$ on $\mathcal{M}$:

$$\phi_{\mathcal{M}}(x+y_1)\cdots\phi_{\mathcal{M}}(x+y_n) \simeq \sum_{(N)} c^{(N)}_{\mathcal{M},x}(y_1,\dots,y_n)\phi^{(N)}_{\mathcal{M}}(x)$$

$\simeq$ is asymptotic equality as $y_j \longrightarrow 0$, valid as expectation values
$\phi^{(N)}_{\mathcal{M}}(x)$ are, eg., suitable Wick-powers of $\phi_{\mathcal{M}}(x)$

Theorem (PCT for OPEs)

Suppose that the $\Phi_{\mathcal{M}}$, $\mathcal{M} \in \mathrm{Obj}(\mathfrak{ManSp})$, induce a local covariant QFT,

- all $\phi_{\mathcal{M}}$ admit OPEs,
- the $c^{(N)}_{\mathcal{M},x}$ are **covariant, fulfil an analytic μSC, and depend analytically on the spacetime metric**.

Then

$$\phi^{C}_{\overline{\mathcal{M}}}(x+y_1)\cdots\phi^{C}_{\overline{\mathcal{M}}}(x+y_n) \simeq \sum_{(N)} \overline{c^{(N)}_{\mathcal{M},x}(y_1,\dots,y_n)}\,\phi^{(N)C}_{\overline{\mathcal{M}}}(x)$$

with $\quad \phi^{(\dots)C}_{\mathcal{M}}(f) = i^F(-1)^S\phi^{(\dots)}_{\mathcal{M}}(\overline{f})$

$F = 0/1$ if $\phi^{(\dots)}_{\mathcal{M}}$ is bosonic/fermionic

$S =$ number of (proper) spinor indices of $\phi^{(\dots)}_{\mathcal{M}}$

Asymptotic form of $\quad V_{pct}\phi_{\mathcal{M}}(f)V_{pct}^{-1} = \phi^{C}_{\overline{\mathcal{M}}}(f)$

III. WICK PRODUCTS AND TIME ORDERED PRODUCTS

Wick products and time ordered products are needed in the perturbative construction of interacting QFTs
(Hollands & Wald, 2001-02; BFV, 2003)

Wick-square of scalar field can be defined with respect to a **reference state** ω:

$$:\varphi^2:_\omega(x) = \lim_{y \to x} \left(\varphi(x)\varphi(y) - \omega(\varphi(x)\varphi(y)) \right)$$

Problem: This definition is **not** generally covariant in the sense that

$$\alpha_\psi(:\varphi^2:_\omega(x')) = (:\varphi^2:_\omega)(\psi(x'))$$

with a suitable algebra-morphism

$$\alpha_\psi : \mathcal{W}(\mathcal{M}') \to \mathcal{W}(\mathcal{M})$$

for $\mathcal{W}(\mathcal{M})$ = algebra generated by free field and Wick polynomials on $\mathcal{M}$ (in quasifree representations fulfilling μSC)

But the problem can be solved by suitable re-definition of the Wick product.

If $\omega, \tilde{\omega}$ are two reference states satisfying μSC, then

$$: \varphi^2 :_\omega (x) - : \varphi^2 :_{\tilde{\omega}} (x) = B_{\omega,\tilde{\omega}}(x)$$

with a smooth function $B_{\omega,\tilde{\omega}}$ satisfying a **covariance condition**

$$B_{\omega\circ\alpha_\psi,\tilde{\omega}\circ\alpha_\psi} = B_{\omega,\tilde{\omega}} \circ \psi$$

and the **cocycle condition**

$$B_{\omega_1,\omega_2} + B_{\omega_2,\omega_3} + B_{\omega_3,\omega_1} = 0$$

This cocycle can be **trivialized;**
For each state $\omega = \omega_{\mathcal{M}'}$ on $\mathscr{A}(\mathcal{M}') = C^*$-algebra of free scalar field on $\mathcal{M}'$ there is $f_{\omega_{\mathcal{M}'}} \in C_0^\infty(\mathcal{M}')$ with

$$f_{\omega\circ\alpha_\psi} = f_\omega \circ \psi \,, \quad B_{\omega,\tilde{\omega}} = f_\omega - f_{\tilde{\omega}}$$

for $\omega = \omega_{\mathcal{M}'}$, $\tilde{\omega} = \tilde{\omega}_{\mathcal{M}}$

Then

$$: \varphi^2 :_{(M,\boldsymbol{g})} (x) = \; : \varphi^2 :_{\omega_{\mathcal{M}}} - f_{\omega_{\mathcal{M}}}(x)$$

is **local and generally covariant** in the above sense.

This establishes existence of locally covariant Wick products of the free field on globally hyperbolic spacetimes. On can also establish the existence of locally covariant time ordered products (Hollands and Wald, 2001-02)

Making natural additional assumptions on μSC, algebraic relations, behaviour under scaling, and continuity, Hollands and Wald have shown that **local covariance fixes Wick-products and time ordered products up to corrections determined by the local curvature.**

E.g. if $:\varphi^k:$ and ${}^{+}_{+}\varphi^{k}{}^{+}_{+}$, $k \in \mathbb{N}$, are two families of Wick products which are generally covariant and fulfill the additional assumptions, then

$${}^{+}_{+}\varphi^{k}{}^{+}_{+}(x) =: \varphi^k : (x) + \sum_{i=0}^{k-2} \binom{k}{i} C_{k-i}(x) \ :\varphi^k:(x)$$

where C_k are polynomials in the metric $\boldsymbol{g}$, the curvature and its derivatives up to order $k-2$. C_k scales homogeneously in λ of degree k under rescalings $\boldsymbol{g} \longrightarrow \lambda^{-2}\boldsymbol{g}$, $m^2 \longrightarrow \lambda^2 m^2$.

A similar assertion holds for the case of the time ordered products (see refs. for details)

Thus, the principle of general covariance is instrumental to reducing the renormalization ambiguity in QFT on manifolds to finitely many renormalization constants (in each order of pert. theory) which can be determined by measurement (since local curvature can be measured) and scale-fixing.

Outlook: Generally covariant QFT over non-commutative spacetimes

Idea: Replace **classical** spacetimes (M, g) by **non-commutative spacetime geometries** and give a similar description of the principle of local, general covariance.

This means:

Replace (M, g) by (A, D, H) : algebraic data of a non-commutative geometry (Connes)

This suggests to consider **generally covariant QFT over non-commutative spacetimes** as a general framework for a theory of quantum gravity (Paschke and Verch, 2004).

$$
\begin{tikzcd}
(A,D,H) & (B,D_B,H_B) \arrow[l, hook'] \arrow[rr, "W"] \arrow[d, "\mathbf{A}_\Phi"'] & & (B',D'_{B'},H'_{B'}) \arrow[r, hook] \arrow[d, "\mathbf{A}_\Phi"] & (A',D',H') \\
\mathbf{A}_\Phi(A,D,H) & \mathbf{A}_\Phi(B,D_B,H_B) \arrow[l, hook'] \arrow[rr, "\alpha_W"] & & \mathbf{A}_\Phi(B',D'_{B'},H'_{B'}) \arrow[r, hook] & \mathbf{A}_\Phi(A',D',H')
\end{tikzcd}
$$

$$\alpha_W \circ \alpha_V = \alpha_{W \circ V}$$

Requires **Lorentzian noncommutative geometry**, i.e. a generalization of the concept of a spectral triple (A, D, H) to "non-commutative globally hyperbolic spacetimes".

This is work in progress...

$\rightarrow$ **see talk by Mario Paschke this afternoon**