Discrete p-Form Electromagnetism as a Chain Field Theory

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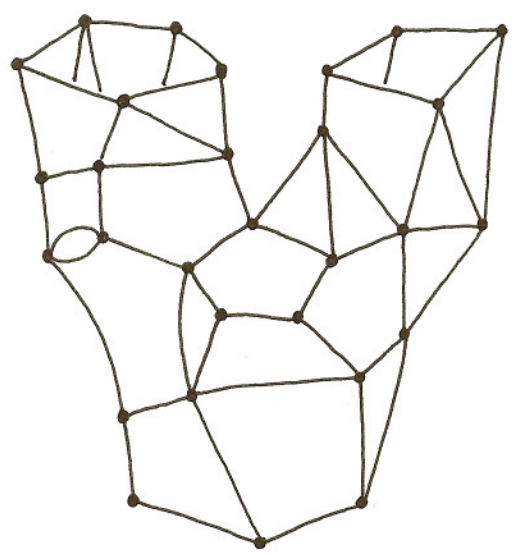

based on gr-qc/0510033

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Generalizing Lattice Gauge Theory

gr-qc/0510033

Usual LGT:

- Approximate Euclidean spacetime by a cubical lattice with

vertices	edges	plaquettes	3-cells	4-cells
– 'events'	– path elts.	– area elts.	– volume elts.	– 4-vol. elts.

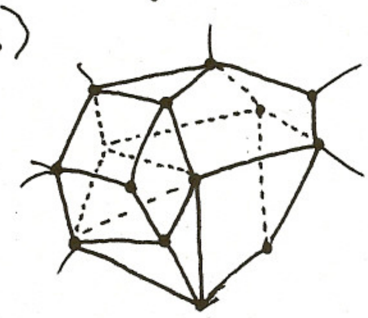
- The gauge field is a map

$$A : [\text{edges}] \longrightarrow G$$

gauge group

Two goals:

- More general discrete spacetimes — E.g. any PLCW complex (as in spin foams)



- Describe 'higher gauge theory' using LGT-like techniques

$$A : [p\text{-cells}] \longrightarrow G$$

In general it's not possible to accomplish both goals in any direct way. When G is <u>abelian</u> we can do it, giving a theory of <u>discrete</u> <u>p-form</u> <u>electromagnetism</u>.

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p-Form Electromagnetism

The gauge field in electromagnetism is a connection on a $U(1)$ bundle, locally a 1-form

$$A = A_\mu dx^\mu$$

Integrating this along a charged particle's worldline gives a term in the action

$$S = \int_\gamma A$$

the connection is flat

$$\Updownarrow$$

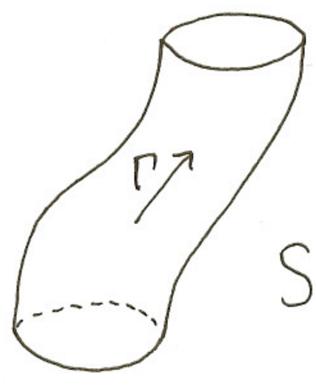
$$F := dA = 0$$

↖ curvature 2-form

In p-form electromagnetism, we generalize this story by promoting A to a p-form

$$A = A_{\mu_1 \mu_2 \cdots \mu_p} \frac{1}{p!} dx^{\mu_1} \wedge dx^{\mu_2} \wedge \cdots \wedge dx^{\mu_p}$$

This interacts naturally not with point particles but with (p-1)-dimensional objects tracing out p-dimensional worldsheets



$$S = \int_\Gamma A$$

the 'p-connection' is flat

$$\Updownarrow$$

$$F := dA = 0$$

↖ curvature (p+1)-form

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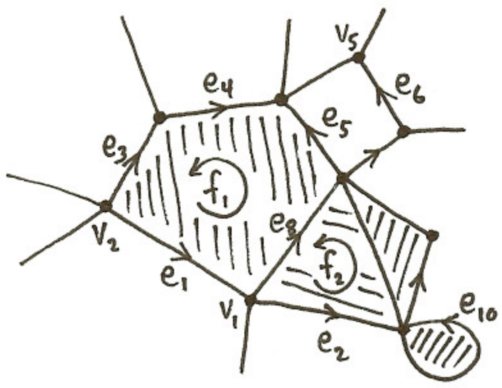
Discrete Spacetime as a Chain Complex

We can model discrete spacetime as some n-dimensional cell complex:



$X_0 = \{v_1, v_2, \ldots\}$ = the set of vertices

$X_1 = \{e_1, e_2, \ldots\}$ = the set of edges

$X_2 = \{f_1, f_2, \ldots\}$ = the set of faces

$\vdots$

X_n = the set of n-cells

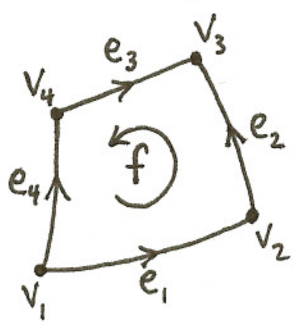
To describe discrete spacetime more algebraically, we let

$$C_k := \text{the free abelian group on the set } X_k \quad \cong \mathbb{Z}^{X_k}$$

be the group of k-chains, and define boundary homomorphisms

$$\partial : C_k \longrightarrow C_{k-1}$$

in the obvious geometric way, e.g.:



has

$$\partial e_1 = v_2 - v_1 \in C_0$$

$$\partial f = e_1 + e_2 - e_3 - e_4 \in C_1$$

and hence

$$\partial\partial f = (v_2 - v_1) + (v_3 - v_2) - (v_3 - v_4) - (v_4 - v_1) = 0 \in C_0$$

The principle that "the boundary of a boundary is zero" says we get a chain complex

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n$$

This is our model of discrete spacetime!

Discrete p-Connections, Curvature, & Gauge Transformations

To get physical fields, we dualize our spacetime chain complex:

$$0 \longleftarrow C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} \cdots \xleftarrow{\partial} C_n$$

to get:

$$C^0 \xrightarrow{d} C^1 \xrightarrow{d} \cdots \xrightarrow{d} C^n \longrightarrow 0$$

where $C^k := \hom(C_k, U(1))$ is the group of $U(1)$-valued k-cochains and the coboundary maps d are defined by $df(c) := f(\partial c)$.

In lattice electromagnetism, the connection assigns to each edge an element of $U(1)$ — essentially the holonomy of the continuum connection. In the p-form generalization, we should get an elt. of $U(1)$ for each p-cell. We thus define a discrete $U(1)$ p-connection to be a homomorphism

$$A : C_p \longrightarrow U(1)$$

so the group of p-connections on the chain complex C is

$$\mathcal{A}(C) := C^p.$$

Similarly, the field strength or curvature is the $(p+1)$-cochain

$$F := dA : C_{p+1} \longrightarrow U(1) \qquad F \in C^{p+1}$$

Two p-connections are gauge equivalent if they differ by a coboundary:

$$A' \sim A \iff A' = A + d\varphi \quad \exists \varphi \in C^{p-1}$$

so we call

$$\mathcal{G}(C) := C^{p-1}$$

the group of gauge transformations.

The Action

When $G = \mathbb{R}$, there is an obvious choice for the action

$$S : C^p \to \mathbb{R}$$

Namely, since $C^{p+1} := \hom(C_{p+1}, \mathbb{R}) \cong \mathbb{R}^{X_{p+1}}$ is a real vector space, we can give it an inner product $\langle -,- \rangle$ and let

$$S(A) := \frac{1}{2e^2} \langle dA, dA \rangle = \frac{1}{2e^2} \langle F, F \rangle$$

analogous to $\frac{1}{2e^2} \int F \wedge \star F$

This leads to Gaussian path integrals

$$Z = \int e^{-S} = \int e^{-\frac{1}{2e^2} \langle F, F \rangle}$$

In the case $G = U(1)$, we define not the action S, but the analog of the Gaussian e^{-S} from the $\mathbb{R}$-theory:

$$e^{-S} = \sum_{n \in \mathbb{Z}^{X_{p+1}}} e^{-\frac{1}{2e^2} \langle \hat{F} - 2\pi n, \hat{F} - 2\pi n \rangle}$$

This is really a theta function.

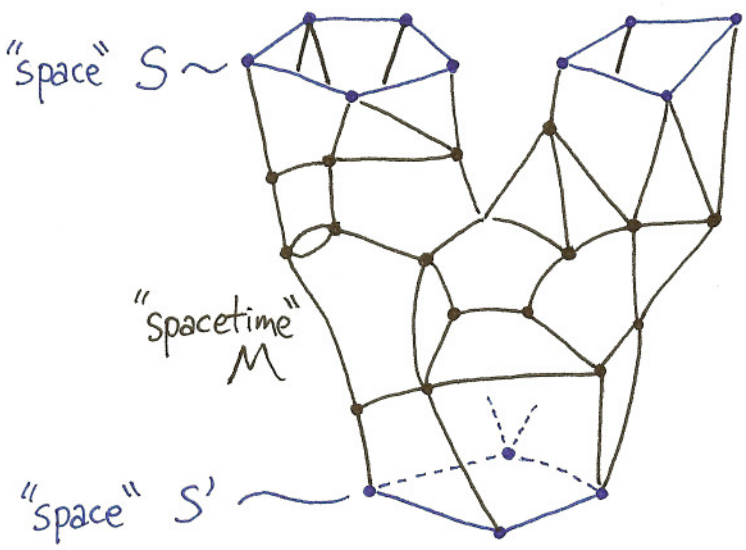
Here

$F = dA \in U(1)^{X_{p+1}}$

and $\hat{F} \in \mathbb{R}^{X_{p+1}}$ is such that $e^{2\pi i \hat{F}_k} = F_k$.

Chain Cobordisms

To describe "time evolution" in discrete p-form electromagnetism, we'd like a notion of spacetime connecting slices of "space":



$$\begin{tikzcd} S_0 \arrow[d, hook] & S_1 \arrow[l] \arrow[d, hook] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d, hook] & \\ M_0 & M_1 \arrow[l] & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\ S'_0 \arrow[u, hook] & S'_1 \arrow[l] \arrow[u, hook] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u, hook] & \end{tikzcd}$$

These form a nice category called nChain:

- objects: chain complexes $S_0 \leftarrow S_1 \leftarrow \cdots \leftarrow S_{n-1}$
- morphisms: chain cobordisms

$$\begin{tikzcd} S_0 \arrow[d] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d] & \\ M_0 & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\ S'_0 \arrow[u] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u] & \end{tikzcd} \quad \text{or} \quad \begin{tikzcd} S \arrow[dr] & \\ & M \\ S' \arrow[ur] & \end{tikzcd}$$

- composition: use 'pushouts':

$$\left(\begin{tikzcd} S \arrow[dr] & \\ & M \\ S' \arrow[ur] & \end{tikzcd} \right)$$

$$\left(\begin{tikzcd} S' \arrow[dr] & \\ & M' \\ S'' \arrow[ur] & \end{tikzcd} \right)$$

$$\Longrightarrow \quad \begin{tikzcd} S \arrow[dr] & & \\ & M \arrow[dr] & \\ S' \arrow[ur] \arrow[dr] & \arrow[r, phantom, "\ulcorner"] & MM' \\ & M' \arrow[ur] & \\ S'' \arrow[ur] & & \end{tikzcd}$$

Chain Field Theory

In analogy to topological quantum field theory, I define a <u>chain field theory</u> to be a symmetric monoidal functor:

$$Z : n\mathrm{Chain} \longrightarrow \mathrm{Hilb}$$

$$\begin{array}{c} S \\ \Big\downarrow M \\ S' \end{array}$$

$$\begin{tikzcd}
S_0 \arrow[d] & \cdots \arrow[l] & S_{n-1} \arrow[l] \arrow[d] & \\
M_0 & \cdots \arrow[l] & M_{n-1} \arrow[l] & M_n \arrow[l] \\
S'_0 \arrow[u] & \cdots \arrow[l] & S'_{n-1} \arrow[l] \arrow[u] &
\end{tikzcd}
\longmapsto
\begin{array}{c} Z(S) \\ \Big\downarrow Z(M) \\ Z(S') \end{array}$$

$Z(S)$ — "Hilbert space of states on S"

$Z(M)$ — linear operator — "time evolution"

$Z(S')$ — "Hilbert space of states on S'"

<u>Theorem</u>: Discrete p-form electromagnetism is a chain field theory, with:

- for each object $S \in n\mathrm{Chain}$,

$$Z(S) = L^2\left(\frac{A(S)}{\mathcal{G}(S)}\right)$$

- for each chain cobordism $M: S \longrightarrow S'$, the time evolution $Z(M): Z(S) \longrightarrow Z(S')$ given by the path integral

$$\langle \psi, Z(M)\phi \rangle = \int_{A(M)} \overline{\psi}(A|_{S'})\,\phi(A|_S)\,e^{-S(A)}\,DA\,.$$

(where DA is a product of $U(1)$-Haar measures, suitably normalized.)